Derivation of O(q^4) Effective Pion-Nucleon Lagrangian Within Heavy Baryon Chiral Perturbation Theory

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Abstract

We construct a complete list of O(q^4) terms directly within Heavy Baryon Chiral Perturbation Theory (HBChPT) in the absence of external fields and assuming isospin symmetry. In addition to a phase rule recently developed [1], a variety of algebraic identities and reparameterization invariance, are used to ensure linear independence of the terms and their low energy coupling constants. We first construct O(q^4) terms for off-shell nucleons, and then perform the on-shell reduction, again within HBChPT. We discuss an application of the O(q^4) terms to the evaluation of the O(q^4) operator insertions in the "contact graph" of pion double charge exchange.

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1 Introduction

Heavy Baryon Chiral Perturbation Theory (HBChPT) is a nonrelativistic (with respect to the "heavy" baryons) effective field theory used for studying meson-baryon interactions at low energies, typically below the mass of the first non-Goldstone resonance (See [2, 3, 4]). The degrees of freedom of SU(2) ($\equiv$ isospin) HBChPT (which will be considered in this paper) are the (derivatives of) pion-triplet and the nucleon fields.

Recently, a method was developed to generate HBChPT Lagrangian ($\mathcal{L}_{\text{HBChPT}}$) for off-shell nucleons *directly within HBChPT*, which as stated in [1], can prove useful when applying HBChPT to nuclear processes in which the nucleons are bound, and hence off-shell. This method has the advantage of not having to bother to start with the relativistic BChPT Lagrangian and then carry out the nonrelativistic reduction. It is thus shorter than the standard approach to HBChPT as given in [5], showed explicitly up to $\mathrm{O}(q^3)$ in [1]. In the context of off-shell nucleons, the upshot of the method developed is a phase rule (See (6),[1]) to implement charge conjugation invariance (along with Lorentz invariance, parity, hermiticity and isospin symmetry) directly within HBChPT. The phase rule, along with additional reductions from a variety of algebraic identities, was used to construct, directly within HBChPT, a complete list of off-shell $\mathrm{O}(q^3)$ terms (in the absence of any external fields and in the isospin-conserving approximation). We also showed that the on-shell limit of the list of terms obtained matches the corresponding list in [5] (in which the HBChPT Lagrangian up to $\mathrm{O}(q^3)$ was constructed starting from the relativistic BChPT Lagrangian). The extension of [1] to $\mathrm{O}(q^4, \phi^{2n})$($\phi \equiv$pion field) was carried out at the time of writing of [1]. We have since extended the work to $\mathrm{O}(q^4, \phi^{2n+1})$ to give the complete $\mathrm{O}(q^4)$ list.

For a complete and precise calculation in the single-nucleon sector to one loop, e.g., 1-loop corrections to pion production off a single (on-shell) nucleon, because of convergence problems (assocated with the amplitude "D_2" for pion production off a single nucleon), one needs to go up to $\mathrm{O}(q^4)$ (See [4, 6, 7]). A complete list of the *divergent* $\mathrm{O}(q^4)$ π-nucleon interaction terms in the presence of external fields was constructed in [7], but again starting from the relativistic BChPT Lagrangian. In this paper, we construct a complete list of off-shell $\mathrm{O}(q^4)$ terms *working entirely within the framework of HBChPT*. We then identify the finite terms at $\mathrm{O}(q^4)$. Even though, unlike [7], we drop the external fields and work in the isospin-conserving approximation, it should be noted that the techniques developed in [1] and this

paper, can be easily extended to include the external fields (in the covariant derivative and as suitable field strengths) as well as include isospin violation.

In addition to the phase rule and a whole new set of algebraic identities (specific to $\mathrm{O}(q^4)$ terms), an additional source for reduction in the number of independent $\mathrm{O}(q^4, \phi^{2n})$ (ϕ is the pion field) low energy coupling constants (LECs) is invariance of $\mathcal{L}_{\mathrm{HBChPT}}$ under infinitesimal variations in the nucleon/baryon velocity, referred to as reparameterization invariance (See Section 5).

Section 2 has the basics and sets up the notations. In Section 3, reductions obtained in addition to (6) due to algebraic identities such as Jacobi(-like) identities, etc. are discussed. In Section 4, the complete lists of $\mathrm{O}(q^4)$ terms is given. In Section 5, further reductions in the number of independent coupling constants due to reparameterization invariance, is discussed. In section 6, we discuss an application of the $\mathrm{O}(q^4)$ list in the multi-nucleon sector. In Section 7 we discuss the derivation of the on-shell $\mathrm{O}(q^4)$ Lagrangian, again within HBChPT using the techniques of [1]. Section 8 has the conclusion which includes remarks on the work done.

2 The Basics

The HBChPT Lagrangian is written in terms of the "upper component" H (and its hermitian adjoint $\bar{\mathrm{H}}$), exponentially parameterized matrix-valued meson fields U, $u \equiv \sqrt{U}$, baryon ("v_μ, S_ν") and pion-field-dependent ("$\mathrm{D}_\mu, u_\nu, \chi_\pm$") building blocks defined below:

$$\mathrm{H} \equiv e^{imv\cdot x}\frac{1}{2}(1+\not{v})\psi, \tag{1}$$

where $\psi \equiv$ Dirac spinor and $m \equiv$ the nucleon mass;

$$\begin{aligned} &v_\mu \equiv \text{nucleon veclocity}, \\ &\mathrm{S}_\nu \equiv \frac{i}{2}\gamma^5\sigma_{\nu\rho}v^\rho \equiv \text{Pauli}-\text{Lubanski spin operator}; \end{aligned} \tag{2}$$

$$U = \exp\left(i\frac{\phi}{F_\pi}\right), \text{ where } \phi \equiv \vec{\pi}\cdot\vec{\tau}, \tag{3}$$

where $\vec{\tau} \in$ nucleon isospin generators; $\mathrm{D}_\mu = \partial_\mu + \Gamma_\mu$ where $\Gamma_\mu \equiv \frac{1}{2}[u^\dagger, \partial_\mu u]$; $u_\mu \equiv i(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger)$; $\chi_\pm \equiv u^\dagger\chi u^\dagger \pm u\chi^\dagger u$, where $\chi \equiv \mathrm{M}_\pi^2$ for this paper.

Terms of the $\mathcal{L}_{\mathrm{(H)BChPT}}$ constructed from products of building blocks will automatically be chiral invariant. Symbolically, a term in $\mathcal{L}_{\mathrm{HBChPT}}$

can be written as just a product of the building blocks to various powers (omitting $\mathrm{H}, \bar{\mathrm{H}}$ as will be done in the rest of the paper except for Section 5):

$$\mathrm{D}_\alpha^m u_\beta^n \chi_+^p \chi_-^q v_\sigma^l \mathrm{S}_\kappa^\mathrm{r} \equiv (\mathrm{m},\mathrm{n},\mathrm{p},\mathrm{q}) \equiv \mathrm{O}(\mathrm{q}^{\mathrm{m+n+2p+2q}}). \tag{4}$$

A systematic path integral derivation for $\mathcal{L}_{\mathrm{HBChPT}}$ based on a paper by Mannel et al [8], starting from $\mathcal{L}_{\mathrm{BChPT}}$ was first given by Bernard et al [6]. As shown by them, after integrating out h from the generating functional, one arrives at $\mathcal{L}_{\mathrm{HBChPT}}$:

$$\mathcal{L}_{\mathrm{HBChPT}} = \bar{\mathrm{H}}\Big(\mathcal{A} + \gamma^0 \mathcal{B}^\dagger \gamma^0 \mathcal{C}^{-1} \mathcal{B}\Big)\mathrm{H}, \tag{5}$$

an expression in the upper components only i.e. for non-relativistic nucleons. So the terms of $\mathcal{L}_{\mathrm{HBChPT}}$ in this paper are given as operators on the H-spinors. For off-shell nucleons, $\gamma^0 \mathcal{B}^\dagger \gamma^0 \mathcal{C}^{-1} \mathcal{B} \in \mathcal{A}$, and hence, listing $\mathcal{A}$-type terms will suffice for this paper.

For the sake of completeness, we will state the phase rule derived in [1]. HBChPT terms (that are Lorentz scalar - isoscalars of even parity) made hermitian using a prescription for constructing hermitian (anti-)commutators discussed in [1], consisting of q $i\chi_-$'s, $P[\ ,\ ]$'s and j (which can take only the values 0 or 1) $\epsilon^{\mu\nu\rho\lambda}$'s for which the following phase rule is satisfied, are the only terms allowed:

$$(-1)^{q+P+j} = 1. \tag{6}$$

In [1], (6) was used to generate complete lists up to $\mathrm{O}(q^3)$ in the absence of external vector and axial-vector fields. In this paper, the same phase rule is used to construct complete lists of $\mathrm{O}(q^4)$, which become relevant in e.g., multi π-production vertices (e.g. $\pi + N \to m\pi + N$) embedded in nuclear processes, when working up to $\mathrm{O}(q^3)$, e.g., for processes involving effectively two nucleons such as pion double charge exchange (DCX) (See [9, 10] and Section 6).

Let A, B, C, D be operators chosen from the pion-field dependent building blocks of (4). In what follows, and especially in Section 3, use will be made of a notation of [11]: $(A, B) \equiv [A, B]$ or $[A, B]_+$. One can then show that apart from the (0,0,0,2) and (0,0,2,0) terms (using the notation of (4)), the following is the complete list of $\mathrm{O}(q^4, \phi^{2n})$ terms (using (6)):

(i) $\quad (A,(B,(C,D))) \equiv (a)[A,[B,[C,D]_+]];\ (b)[A,[B,[C,D]]_+];$
$\quad (c)[A,[B,[C,D]]]_+;\ (d)[A,[B,[C,D]_+]_+]_+;$

(ii) $\quad ((A,B),(C,D)) \equiv (a)[[A,B],[C,D]_+];\ (b)[[A,B]_+,[C,D]];$

$$
\begin{array}{ll}
 & (c)[[A,B],[C,D]]_+;\ (d)[[A,B]_+,[C,D]_+]_+; \\
(iii) & i(A,(B,(C,D))) \equiv (a)i[A,[B,[C,D]_+]_+];\ (b)i[A,[B,[C,D]_+]]_+; \\
 & (c)i[A,[B,[C,D]]_+]_+;\ (d)i[A,[B,[C,D]]]; \\
(iv) & i((A,B),(C,D)) \equiv (a)i[[A,B],[C,D]];\ (b)i[[A,B]_+,[C,D]_+]; \\
 & (c)i[[A,B],[C,D]_+]_+;\ (d)i[[A,B]_+,[C,D]]_+; \\
(v) & i(A,(B,C)) \equiv (a)i[A,[B,C]]_+;\ (b)i[A,[B,C]_+];\ (c)A \leftrightarrow B; \\
 & (d)i[[A,B],C]_+;\ (e)i[[A,B]_+,C]; \\
(vi) & (A,(B,C)) \equiv (a)[A,[B,C]];\ (b)[A,[B,C]_+]_+;\ (c)A \leftrightarrow B \\
 & (d)[[A,B],C];\ (e)[[A,B]_+,C]_+,
\end{array}
\tag{7}
$$

where it is understood that of all the possible terms implied by $(i)(A,(B,(C,D)))$, $(i)((A,B),(C,D))$ and $(i)(A,(B,C))$, only those that are allowed by (6) are to be included. For (0,0,2,0)- and (0,0,0,2)- type terms, one needs to include

$$
(A,A) \equiv (\chi_+,\chi_+);\ (\chi_-,\chi_-). \tag{8}
$$

The list (7) holds good for $\mathrm{O}(q^4,\phi^{2n+1})$ terms with the difference that there is an additional factor of i multiplying the terms in $(i),(ii)$ and (vi), and the i in $(iii),(iv)$ and (v), is absent. The reason for including i only in some combinations of terms has to do with imposing charge conjugation invariance along with other symmetries *directly within HBChPT* (See [1]). The terms of (7) and its analog for $\mathrm{O}(q^4,\phi^{2n+1})$ are not all independent since they can be related by a number of linear relations: see next section (and [1] for $\mathrm{O}(q^3)$).

3 Further Reduction due to Algebraic identities

In this section, we discuss further reduction in addition to the ones obtained from (6). The main result from [1] is that one need not consider trace-dependent terms in SU(2) HBChPT if one assumes isospin conservation. Thus, trace-dependent $\mathrm{O}(q^4)$ terms can be eliminated in preference for trace-independent terms. We discuss reduction due to algebraic identities in the various categories of (7). Some of the algebraic reductions require one to consider more than one category at a time, e.g., for $\mathrm{O}(q^4,\phi^{2n})$ terms, the Jacobi-like identities in (12) require one to consider $(i),(i)(A\leftrightarrow B),(ii)$. After (9)[curvature relation], we discuss the algebraic reductions in $\mathrm{O}(q^4,\phi^{2n})$ and $\mathrm{O}(q^4,\phi^{2n+1})$ terms, separately.

In the absence of external vector and axial-vector fields, one can show that:

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = \frac{1}{4}[u_\mu, u_\nu], \tag{9}$$

which is referred to as the "curvature relation." This relation will be used extensively in conjunction with some Jacobi-like identities discussed below. It is because of this identity that one requires to consider some (4,0,0,0), (0,4,0,0) and (2,2,0,0) terms together, e.g., in (13).

3.1 O(q^4, ϕ^{2n}) Terms

In this subsection, we consider reduction in the number of independent O(q^4, ϕ^{2n}) terms due to various algebraic identities. The following are the algebraic identities responsible for reduction in number of O(q^4, ϕ^{2n}) terms: (10), (11), (12), (18), (27), (28), (30) and (31). For (12) and (18), there are two sets each of terms (one $\epsilon^{\mu\nu\rho\lambda}$-dependent and the other $\epsilon^{\mu\nu\rho\lambda}$-independent), that get eliminated. One of the two sets ($\epsilon^{\mu\nu\rho\lambda}$- independent) for (12) has been discussed in detail in this subsection. The details for the other sets are given as appendices.

$\underline{\underline{p = q = 0 \text{ in } (4) \equiv (A, (B, (C, D))); ((A, B), (C, D))}}$

This includes $(i) - (iv)$ of (7). All terms in each of the first four types (of terms) in (7) $[(i) - (iv)]$ are linearly independent for unequal field operators A, B, C, D. However for (4,0,0,0), (0,4,0,0) and (2,2,0,0), L.C.-independent terms, one needs to consider A=C, B=D in (i) in equation (7). Using

$$[A, [B, [A, B]_+]] = -[A, [B, [A, B]]_+] \tag{10}$$

only three of the four terms in (i) of equation (7), are linearly independent. Similarly, using

$$[[A, B], [A, B]_+] = -[[A, B]_+, [A, B]], \tag{11}$$

only three of the four terms in (ii) of equation (7), are linearly independent.

There are some reductions possible due to some Jacobi-like identities by considering : $(i), (i)(A \leftrightarrow B), (ii)$ of $(7)(\equiv \epsilon^{\mu\nu\rho\lambda}$ -independent terms), and $(iii), (iii)(A \leftrightarrow B), (iv)$ of (7) ($\equiv \epsilon^{\mu\nu\rho\lambda}$-dependent terms). The reason why one can not hope to get reductions by considering any other pairs of types of terms in $(i) - (iv)$ (in (7)), is because one can get (linear) algebraic relationships only between those terms which are (both) independent of (have) an overall factor of i.

$\underline{(i),(i)(A\leftrightarrow B),(ii) \text{ of } (7)}$

One can show the following 8 Jacobi-like identities:

$$
\begin{aligned}
&[A,[B,[C,D]_+]]-[[A,B],[C,D]_+]=(i)(a)(A\leftrightarrow B)\\
&[A,[B,[C,D]_+]]-[[A,B]_+,[C,D]_+]_+=-(i)(d)(A\leftrightarrow B)\\
&[A,[B,[C,D]]_+]-[[A,B]_+,[C,D]]=-(i)(b)(A\leftrightarrow B)\\
&[A,[B,[C,D]]_+]-[[A,B],[C,D]]_+=(i)(c)(A\leftrightarrow B)\\
&[A,[B,[C,D]]]_+-[[A,B]_+,[C,D]]=-(i)(c)(A\leftrightarrow B)\\
&[A,[B,[C,D]_+]_+]_+-[[A,B],[C,D]_+]=(i)(d)(A\leftrightarrow B)
\end{aligned}
\tag{12}
$$

Since we have 6 identities in 12 terms, we can take any 6 as linearly independent, say $(i)\Big((a)-(d)\Big)$ and $(ii)\Big((a),(b)\Big)$ of (7). Obviously, to ensure linear independence, one can not choose these four terms such that any three belong to the same (Jacobi-like)identity

(1) Using (12) and (9), one needs to consider the following (4,0,0,0), (0,4,0,0) and (2,2,0,0) $\epsilon^{\mu\nu\rho\lambda}$-independent terms together:

$$
\begin{aligned}
&(v\cdot \mathrm{D},(\mathrm{D}_\mu,(v\cdot \mathrm{D},\mathrm{D}^\mu))),\ (\mathrm{D}_\mu,(v\cdot \mathrm{D},(v\cdot \mathrm{D},\mathrm{D}^\mu))),\ ((v\cdot \mathrm{D},\mathrm{D}_\mu),(v\cdot \mathrm{D},\mathrm{D}^\mu))\\
&(v\cdot u,(u_\mu,(v\cdot u,u^\mu))),\ (u_\mu,(v\cdot u,(v\cdot u,u^\mu))),\ ((v\cdot u,u_\mu),(v\cdot u,u^\mu)),\\
&(v\cdot \mathrm{D},(\mathrm{D}_\mu,(v\cdot u,u^\mu))),\ (\mathrm{D}_\mu,(v\cdot \mathrm{D},(u_\mu,v\cdot u))),\ ((v\cdot \mathrm{D},\mathrm{D}_\mu),(v\cdot u,u^\mu)))\\
&(v\cdot u,(u_\mu,(v\cdot \mathrm{D},\mathrm{D}^\mu))),\ (u_\mu,(v\cdot u,(v\cdot \mathrm{D},\mathrm{D}^\mu))).
\end{aligned}
\tag{13}
$$

One needs to do a careful counting of the total number of identities that one can write down using (12) and (9), and the total number of terms in those identities. We do the same below.

Using (10), (11), (9) and (12), one sees that one gets

(a) 14 identities in 21 terms:

$$
\begin{aligned}
&[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]-\frac{1}{4}[[v\cdot u,u^\mu],[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]=[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]\\
&[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]-[[v\cdot \mathrm{D},\mathrm{D}^\mu]_+,[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]_+=-[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]_+]_+\\
&-[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]-\frac{1}{16}[[v\cdot u,u^\mu],[v\cdot u,u_\mu]]_+=\frac{1}{4}[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot u,u^\mu]]]_+\\
&\frac{1}{4}[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot u,u^\mu]]]_++\frac{1}{4}[[v\cdot u,u^\mu],[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]=-\frac{1}{4}[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot u,u^\mu]]]_+\\
&[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]_+]_+-[[v\cdot u,u^\mu],[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]=[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]_+]_+
\end{aligned}
$$

$$
\begin{aligned}
&[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot u,u^\mu]_+]]-[[v\cdot u,u^\mu]_+,[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]_+=-[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot u,u^\mu]_+]_+]_+\\
&[v\cdot u,[u_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]-[[v\cdot u,u^\mu],[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]=[u_\mu,[v\cdot u,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]\\
&\frac{1}{4}[v\cdot u,[u_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]]-[[v\cdot u,u^\mu],[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]=[u_\mu,[v\cdot u,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]_+]_+\\
&[v\cdot u,[u_\mu,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]_+]_+-[[v\cdot u,u^\mu],[v\cdot \mathrm{D},\mathrm{D}_\mu]_+]=[u_\mu,[v\cdot u,[v\cdot \mathrm{D},\mathrm{D}^\mu]_+]_+]_+\\
&[v\cdot u,[u_\mu,[v\cdot u,u^\mu]_+]]-[[v\cdot u,u_\mu],[v\cdot u,u^\mu]_+]=[u_\mu,[v\cdot u,[v\cdot u,u^\mu]_+]]\\
&[v\cdot u,[u_\mu,[v\cdot u,u^\mu]_+]]-[[v\cdot u,u_\mu]_+,[v\cdot u,u^\mu]_+]_+=[u_\mu,[v\cdot u,[v\cdot u,u^\mu]_+]_+]_+\\
&[v\cdot u,[u_\mu,[v\cdot u,u^\mu]_+]]-[[v\cdot u,u_\mu],[v\cdot u,u^\mu]]_+=[u_\mu,[v\cdot u,[v\cdot u,u^\mu]]]_+\\
&[v\cdot u,[u_\mu,[v\cdot u,u^\mu]]]_+-[[v\cdot u,u_\mu]_+,[v\cdot u,u^\mu]]=[u_\mu,[v\cdot u,[v\cdot u,u^\mu]]]_+\\
&[v\cdot u,[u_\mu,[v\cdot u,u^\mu]_+]_+]_+-[[v\cdot u,u_\mu],[v\cdot u,u^\mu]_+]=[u_\mu,[v\cdot u,[v\cdot u,u^\mu]_+]_+]_+,
\end{aligned}
\tag{14}
$$

and

(b) two identities in five terms:

$$
\begin{aligned}
&[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot u,u^\mu]_+]]-\frac{1}{4}[[v\cdot u,u^\mu],[v\cdot u,u_\mu]_+]=[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot u,u^\mu]_+]]\\
&[v\cdot \mathrm{D},[\mathrm{D}_\mu,[v\cdot u,u^\mu]_+]_+]_+-\frac{1}{4}[[v\cdot u,u^\mu],[v\cdot u,u_\mu]_+]=[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot u,u^\mu]_+]_+]_+.
\end{aligned}
\tag{15}
$$

The reason for considering (14) and (15) separately is because the terms contained in them do not mix. One can thus take $i=2,3,4,19,20,36,37$ and $i=28,44,45$ of Table 1 as the two sets of linearly independent terms.

(2) Similarly, one will need to consider the following (4,0,0,0), (0,4,0,0) and (2,2,0,0) - type terms together:

$$
\begin{aligned}
&(\mathrm{D}_\nu,(\mathrm{D}_\mu,(\mathrm{D}^\nu,\mathrm{D}^\mu))),\ ((\mathrm{D}_\nu,\mathrm{D}_\mu),(\mathrm{D}^\nu,\mathrm{D}^\mu))\\
&(u_\nu,(u_\mu,(u^\nu,u^\mu))),\ ((u_\nu,u_\mu),(u^\nu,u^\mu)),\\
&(\mathrm{D}_\nu,(\mathrm{D}_\mu,(u^\nu,u^\mu))),\ ((\mathrm{D}_\nu,\mathrm{D}_\mu),(u^\nu,u^\mu)))\\
&(u_\nu,(u_\mu,(\mathrm{D}^\nu,\mathrm{D}^\mu))).
\end{aligned}
\tag{16}
$$

As is shown in Appendix A, algebraic identities based on (12) and the curvature relation (9), can be used to select $i=13,14,25,26,28$ and $i=43,44,45$ of Table 1, as the two sets of linearly independent terms.

For the (2,2,0,0)-type terms, using (12), the following result will be used for constructing complete lists of $\mathrm{O}(q^4)$ terms. Using (18), one gets 6+6=12

identities in 12+8=20 terms considered as following triplets:

$$\begin{aligned}&(v\cdot \mathrm{D},((\mathrm{D}_\mu,v\cdot u),u^\mu)),\ (u^\mu,((\mathrm{D}_\mu,v\cdot u),v\cdot \mathrm{D}),\ ((v\cdot \mathrm{D},u^\mu),(\mathrm{D}_\mu,v\cdot u)),\\&(\mathrm{D}_\mu,((v\cdot \mathrm{D},u^\mu),v\cdot u)),\ (v\cdot u,((v\cdot \mathrm{D},u^\mu),\mathrm{D}_\mu)),\end{aligned} \tag{17}$$

implying that one can take eight linearly independent terms, say $i = 71, .., 78$.

<u>$(iii), (iii)(A \leftrightarrow B)$ and (iv) of (7)</u>

One can show the following Jacobi-like identities to be true:

$$\begin{aligned}&i[A,[B,[C,D]_+]_+] - i[[A,B]_+,[C,D]_+] = -(iii)(a)(A\leftrightarrow B)\\&i[A,[B,[C,D]_+]_+] - i[[A,B],[C,D]_+]_+ = (iii)(b)(A\leftrightarrow B)\\&i[A,[B,[C,D]_+]]_+ - i[[A,B]_+,[C,D]_+] = -(iii)(b)(A\leftrightarrow B)\\&i[A,[B,[C,D]]_+]_+ - i[[A,B],[C,D]] = (iii)(c)(A\leftrightarrow B)\\&i[A,[B,[C,D]]_+]_+ - i[[A,B]_+,[C,D]_+] = -(iii)(d)(A\leftrightarrow B)\\&i[A,[B,[C,D]]] - i[[A,B],[C,D]] = (iii)(d)(A\leftrightarrow B)\end{aligned} \tag{18}$$

Again we have 6 identities in 12 terms, implying one can take 6 as linearly independent, say $(iii)\Big((a)-(d)\Big)$ and $(iv)\Big((a),(b)\Big)$ of (7).

(3) The identities (18) along with the curvature relation, require one to consider the following category of $\epsilon^{\mu\nu\rho\lambda}$-dependent terms of the type (4,0,0,0), (0,4,0,0) and (2,2,0,0) terms together:

$$\begin{aligned}&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big[(\mathrm{D}_\mu,(\mathrm{D}_\nu,(\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D}))),\ (u_\mu,(u_\nu,(u_\lambda,\mathrm{S}\cdot u))),\\&(\mathrm{S}\cdot \mathrm{D},(\mathrm{D}_\mu,(\mathrm{D}_\nu,[\mathrm{D}_\nu,\mathrm{D}_\lambda]))\ (\mathrm{D}_\mu,(\mathrm{S}\cdot \mathrm{D},[\mathrm{D}_\nu,\mathrm{D}_\lambda])),\\&(\mathrm{S}\cdot u,(u_\mu,([u_\nu,u_\lambda])),\ (u_\mu,(\mathrm{S}\cdot u,[u_\nu,u_\lambda])),\\&([u_\mu,u_\nu],(\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D})),\ (u_\mu,(u_\nu,(\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D}))),\\&(\mathrm{D}_\mu,(\mathrm{S}\cdot \mathrm{D},[u_\nu,u_\lambda])),\ (\mathrm{S}\cdot \mathrm{D},(\mathrm{D}_\mu,[u_\nu,u_\lambda])),\\&(u_\mu,(\mathrm{S}\cdot u,[\mathrm{D}_\nu,\mathrm{D}_\lambda])),\ (\mathrm{S}\cdot u,(u_\mu,[\mathrm{D}_\nu,\mathrm{D}_\lambda])),\\&(\mathrm{D}_\mu,(\mathrm{D}_\nu,(u_\lambda,\mathrm{S}\cdot u))),\ ([\mathrm{D}_\mu,\mathrm{D}_\nu],(u_\lambda,\mathrm{S}\cdot u))\Big].\end{aligned} \tag{19}$$

One will need to use the following four equations in addition:

$$\epsilon^{\mu\nu\rho\lambda}\Big([[\mathrm{D}_\mu,\mathrm{D}_\nu]_+,[\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D}]]_+ = [[\mathrm{D}_\mu,\mathrm{D}_\nu]_+,[\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D}]_+]\Big) = 0, \tag{20}$$

$$\epsilon^{\mu\nu\rho\lambda}\Big[(\mathrm{D}_\mu,(\mathrm{D}_\nu,(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}))=-(\mathrm{D}_\nu,(\mathrm{D}_\mu,(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}))\Big], \tag{21}$$

$$i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([[u_\mu,u_\nu]_+,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]]_+=[[u_\mu,u_\nu]_+,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]]_+=0\Big), \tag{22}$$

$$i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big((u_\mu,(u_\nu,(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}))=-(u_\nu,(u_\mu,(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}))\Big). \tag{23}$$

Analogous to (13) and (16), one needs to do a careful counting of the total number of identities that one can write down using (18), (9) and (20)-(23), and the total number of terms in those identities. We do the same in Appendix B.

A similar analysis can be carried out for terms with $v\leftrightarrow\mathrm{S}$ in (19).

(4) The identities (18) along with the curvature relation, require one to consider the following category of $\epsilon^{\mu\nu\rho\lambda}$- dependent terms of the type (4,0,0,0), (0,4,0,0) and (2,2,0,0) terms together:

$$\begin{aligned}&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\Big((u_\kappa,(u_\mu,(u^\kappa,u_\nu))),\ (u_\mu,(u_\kappa,(u^\kappa,u_\nu))),\ ((u_\mu,u_\kappa),(u_\nu,u^\kappa)),\\&(\mathrm{D}_\kappa,(\mathrm{D}_\mu,(\mathrm{D}^\kappa,\mathrm{D}_\nu))),\ ((\mathrm{D}_\mu,(\mathrm{D}_\kappa,(\mathrm{D}^\kappa,\mathrm{D}_\nu))),\ ((\mathrm{D}_\kappa,\mathrm{D}_\mu),(\mathrm{D}^\kappa,\mathrm{D}_\nu)),\\&((\mathrm{D}_\kappa,(\mathrm{D}_\mu,(u^\kappa,u_\nu))),\ ((\mathrm{D}_\mu,(\mathrm{D}_\kappa,(u^\kappa,u_\nu))),\ ((\mathrm{D}_\kappa,\mathrm{D}_\mu),(u^\kappa,u_\nu)),\\&(u_\mu,(u_\kappa,(\mathrm{D}^\kappa,\mathrm{D}_\nu))),\ (u_\mu,(u_\kappa,(\mathrm{D}^\kappa,\mathrm{D}_\nu)))\Big).\end{aligned} \tag{24}$$

As is shown in Appendix C, algebraic identities based upon (18) and the curvature relation (9), can be used to select a set of linearly independent terms from from (24). This is done in Appendix C.

A similar table can be constructed with $(u_\kappa,\mathrm{D}^\kappa)\rightarrow(v\cdot u,v\cdot\mathrm{D})$ in (24).

For the (2,2,0,0)-type terms, using (18), the following results will be used for constructing complete lists of $\mathrm{O}(q^4)$ terms.

(a) The following set of 12+8 [1] =20 terms need to be considered together as the following triplets:

$$\begin{aligned}&(i)\quad iv_\rho\epsilon^{\mu\nu\rho\lambda}\Big((\mathrm{D}_\mu,((\mathrm{D}_\nu,u_\lambda),\mathrm{S}\cdot u)),\ (\mathrm{S}\cdot u,((\mathrm{D}_\nu,u_\lambda),\mathrm{D}_\mu));\\&\qquad ((\mathrm{D}_\mu,\mathrm{S}\cdot u),(\mathrm{D}_\nu,u_\lambda));\\&(ii)\quad (\mathrm{D}_\mu,((\mathrm{D}_\nu,\mathrm{S}\cdot u),u_\lambda)),\ (u_\lambda,((\mathrm{D}_\nu,\mathrm{S}\cdot u),\mathrm{D}_\mu)),\\&\qquad ((\mathrm{D}_\mu,\mathrm{S}\cdot u),(\mathrm{D}_\nu,u_\lambda))\Big).\end{aligned} \tag{25}$$

[1]The 8 is because $i\epsilon^{\mu\nu\rho\lambda}v_\rho((\mathrm{D}_\mu,\mathrm{S}\cdot u),(\mathrm{D}_\nu,u_\lambda))$ is common to both (i) and (ii) in (25).

Using (18), one can eliminate all except eight, say: $i = 99, ..., 106$ of Table 1. A similar analysis is carried out to select $i = 77, .., 84; 91, .., 98; 107, .., 114;$ $115, .., 118, 120, 121, 123\ ; 122, 124, ..130; 143, .., 150; 151, .., 158; 159, .., 166;$ $167, .., 174$ of Table 1.

(b) Using (18), one gets 6 identities in 12 terms considered as following triplet:

$$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big((\mathrm{D}_\mu, ((\mathrm{D}_\nu, u_\kappa), u^\kappa)),\ ((\mathrm{D}_\mu, u^\kappa), (\mathrm{D}_\nu, u^\kappa)),$$
$$(u^\kappa, ((\mathrm{D}_\mu, u^\kappa), \mathrm{D}_\nu))\Big), \tag{26}$$

implying that one can take six linearly independent terms, say $i = 131, .., 136$ of Table 1. A similar analysis is carried out to select $i = 71, .., 76; 85, .., 90;$ $137, ..142; 175, .., 180;\ 181, .., 186$ of Table 1.

For (0,4,0,0)-type terms, writing $u_\mu = u^a_\mu \tau^a$, one will need to consider the following reductions:

$$\begin{array}{ll}
(i) & [\tau^a, [\tau^b, [\tau^c, \tau^d]_+]] = [\tau^a, [\tau^b, [\tau^c, \tau^d]]_+] = 0; \\
 & [\tau^a, [\tau^b, [\tau^c, \tau^d]]]_+,\ [\tau^a, [\tau^b, [\tau^c, \tau^d]_+]_+]_+ \neq 0; \\
(ii) & [[\tau^a, \tau^b], [\tau^c, \tau^d]_+] = 0;\ [[\tau^a, \tau^b], [\tau^c, \tau^d]]_+,\ [[\tau^a, \tau^b]_+, [\tau^c, \tau^d]_+]_+ \neq 0; \\
(iii) & i[\tau^a, [\tau^b, [\tau^c, \tau^d]_+]]_+ = 0; \\
 & i[\tau^a, [\tau^b, [\tau^c, \tau^d]]_+]_+,\ i[\tau^a, [\tau^b, [\tau^c, \tau^d]_+]_+],\ i[\tau^a, [\tau^b, [\tau^c, \tau^d]]] \neq 0; \\
(iv) & i[[\tau^a, \tau^b]_+, [\tau^c, \tau^d]_+] = 0;\ i[[\tau^a, \tau^b], [\tau^c, \tau^d]_+]_+,\ i[[\tau^a, \tau^b], [\tau^c, \tau^d]] \neq 0.
\end{array} \tag{27}$$

$p \neq 0$ or $q \neq 0$ in (4)$\equiv (A, (B, C))$

This includes (v) and (vi) of (7).

(v) of (7)

By using the following three Jacobi-like identities which are generalized Jacobi identities as used in graded Lie algebra in supersymmetric theories

$$\begin{array}{l}
i[A, [B, C]]_+ - i[[A, B], C]_+ = i[B, [A, C]_+] \\
i[A, [B, C]]_+ - i[[A, B]_+, C] = -i[B, [A, C]]_+ \\
i[A, [B, C]_+] - i[[A, B]_+, C] = -i[B, [A, C]_+],
\end{array} \tag{28}$$

one sees that one needs to consider only three of the six terms that figure in the above three identities, say $i[A, [B, C]_+], i[A, [B, C]]_+$ and $i[B, [A, C]_+]$ as linearly independent terms. These three identities are similar to the ones

that occur in SUSY graded Lie algebra for $A, C \equiv$ fermionic and $B \equiv$ bosonic fields, $A, B \equiv$ fermionic and $C \equiv$ bosonic fields, and $A, B, C \equiv$ fermionic fields, respectively. The identities in (28) are used in, e.g., $\epsilon^{\mu\nu\rho\lambda}$-independent (1,1,0,1)-type terms. When applying (28) to (2,0,1,0), because of (9), one will need to consider the following terms together:

$$
\begin{aligned}
& i\epsilon^{\mu\nu\rho\lambda} v_\rho \mathrm{S}_\lambda \Bigl((\mathrm{D}_\mu, (\mathrm{D}_\nu, \chi_+)),\ ([\mathrm{D}_\mu, \mathrm{D}_\nu], \chi_+); \\
& (u_\mu, (u_\nu, \chi_+)),\ ([u_\mu, u_\nu], \chi_+) \Bigr).
\end{aligned} \tag{29}
$$

Further noting that u_μ is an isovector and χ_+ is an isoscalar, one sees that:

$$
[u_\mu, \chi_+] = 0. \tag{30}
$$

Applying (28) and (30) to (29), one sees that one can take two linearly independent terms, say $i\epsilon^{\mu\nu\rho\lambda} v_\rho \mathrm{S}_\lambda \Bigl([\mathrm{D}_\mu, [\mathrm{D}_\nu, \chi_+]]_+, [[u_\mu, u_\nu], \chi_+]_+ \Bigr)$.

<u>*(vi)* of (7)</u>

By considering the following three Jacobi(-like) identities :

$$
\begin{aligned}
& [A, [B, C]] - [[A, B], C] = [B, [A, C]] \\
& [A, [B, C]_+]_+ - [[A, B]_+, C]_+ = -[B, [A, C]] \\
& [A, [B, C]] - [[A, B]_+, C]_+ = -[B, [A, C]_+]_+,
\end{aligned} \tag{31}
$$

one needs to consider only three of the six terms, say $[A, [B, C]], [A, [B, C]_+]_+$ and $[B, [A, C]]$. The identities in (31) are used in, e.g., $\epsilon^{\mu\nu\rho\lambda}$-dependent (1,1,0,1)-type terms.

3.2 $\mathrm{O}(q^4, \phi^{2n+1})$ Terms

In this subsection, we consider the reduction in the number of $\mathrm{O}(q^4, \phi^{2n+1})$ terms because of algebraic identities. The discussion in this subsection will be much briefer than the preceding (subsection).

<u>*(i)* − *(iv)* of (7)′</u>

(1) The identities (12) are the same for $\mathrm{O}(q^4, \phi^{2n+1})$ except for an overall factor of i. We will denote the analogue of (12) for $\mathrm{O}(q^4, \phi^{2n+1})$ terms as (12)′.[2] Using it together with (9), one sees that one needs to consider the

[2]Similarly, the analogs of (7), (18), (28) and (31) will be denoted by (7)′, (18)′, (28)′ and (31)′.

following set of terms together:

$$
\begin{aligned}
&i(v\cdot \mathrm{D},(\mathrm{D}_\mu,(\mathrm{D}^\mu,\mathrm{S}\cdot u))),\ i(\mathrm{D}_\mu,(v\cdot \mathrm{D},(\mathrm{D}^\mu,\mathrm{S}\cdot u))),\ i((v\cdot \mathrm{D},\mathrm{D}_\mu),(\mathrm{D}^\mu,\mathrm{S}\cdot u)),\\
&i(\mathrm{S}\cdot u,(\mathrm{D}_\mu,(\mathrm{D}^\mu,v\cdot \mathrm{D}))),\ i(\mathrm{D}_\mu,(\mathrm{S}\cdot u,(\mathrm{D}^\mu,v\cdot \mathrm{D})));\\
&i(v\cdot u,(u_\mu,(\mathrm{S}\cdot u,\mathrm{D}^\mu))),\ i(u_\mu,(v\cdot u,(\mathrm{S}\cdot u,\mathrm{D}^\mu))),\ i((v\cdot u,u_\mu),(\mathrm{S}\cdot u,\mathrm{D}^\mu)),\\
&i(\mathrm{D}_\mu,(\mathrm{S}\cdot u,(u^\mu,v\cdot u)))\ i(\mathrm{S}\cdot u,(\mathrm{D}_\mu,(u^\mu,v\cdot u)));\\
&i(\mathrm{D}_\mu,(\mathrm{D}^\mu,(v\cdot \mathrm{D},\mathrm{S}\cdot u))),\ i(\mathrm{D}^2,(v\cdot \mathrm{D},\mathrm{S}\cdot u)),\\
&i(\mathrm{S}\cdot u,(v\cdot \mathrm{D},\mathrm{D}^2)),\ i(v\cdot \mathrm{D},(\mathrm{S}\cdot u,\mathrm{D}^2)).
\end{aligned}
\tag{32}
$$

One can show that of the terms listed in (32), one can take $i = 208, 209, 210, 245, .., 252, 300$ of Table 2 as a set of linearly independent terms.

Similarly, using $(12)'$ and (9), one can show that of

$$
\begin{aligned}
(a)\quad &i(v\cdot \mathrm{D},(\mathrm{D}_\mu,(\mathrm{S}\cdot \mathrm{D},u^\mu))),\ i(\mathrm{D}_\mu,(v\cdot \mathrm{D},(\mathrm{S}\cdot \mathrm{D},u^\mu))),\ i((v\cdot \mathrm{D},\mathrm{D}_\mu),(\mathrm{S}\cdot \mathrm{D},u^\mu)),\\
&i(u_\mu,(\mathrm{S}\cdot \mathrm{D},(\mathrm{D}^\mu,v\cdot \mathrm{D})))\ i(\mathrm{S}\cdot \mathrm{D},(u_\mu,(\mathrm{D}^\mu,v\cdot \mathrm{D})));\\
&i(v\cdot u,(u_\mu,(u^\mu,\mathrm{S}\cdot \mathrm{D}))),\ i(u_\mu,(v\cdot u,(u^\mu,\mathrm{S}\cdot \mathrm{D}))),\ i((v\cdot u,u_\mu),(u^\mu,\mathrm{S}\cdot \mathrm{D})),\\
&i(\mathrm{S}\cdot \mathrm{D},(u_\mu,(u^\mu,v\cdot u))),\ i(u_\mu,(\mathrm{S}\cdot \mathrm{D},(u^\mu,v\cdot u)))\\
&i(u^\mu,(u_\mu,(v\cdot u,\mathrm{D}^\mu))),\ i(\mathrm{S}\cdot u,(u_\mu,v\cdot u,\mathrm{D}^\mu)),\\
&i((u_\mu,\mathrm{S}\cdot u),(v\cdot u,\mathrm{D}^\mu)),\ i(\mathrm{D}_\mu,(v\cdot u,(u^\mu,\mathrm{S}\cdot u))),\\
&i(v\cdot u,(\mathrm{D}_\mu,(u^\mu,\mathrm{S}\cdot u)));\\
&i(u^\mu,(u_\mu,(v\cdot u,\mathrm{S}\cdot \mathrm{D}))),\ i(u^2,(v\cdot u,\mathrm{S}\cdot \mathrm{D})),\\
&i(\mathrm{S}\cdot \mathrm{D},(v\cdot u,u^2)),\ i(v\cdot u,(\mathrm{S}\cdot \mathrm{D},u^2)).
\end{aligned}
\tag{33}
$$

$i = 213, 214, 244, 261, .., 267, 268, 315$ of Table 2 form a set of linearly independent terms;

$$
\begin{aligned}
(b)\quad &i(v\cdot \mathrm{D},(\mathrm{S}\cdot \mathrm{D},(\mathrm{D}^\mu,u_\mu))),\ i(\mathrm{S}\cdot \mathrm{D},(v\cdot \mathrm{D},(\mathrm{D}^\mu,u_\mu))),\ i((v\cdot \mathrm{D},\mathrm{S}\cdot \mathrm{D}),(\mathrm{D}^\mu,u_\mu)),\\
&i(u_\mu,(\mathrm{D}^\mu,(v\cdot \mathrm{D},\mathrm{S}\cdot \mathrm{D}))),\ i(\mathrm{D}_\mu,(u^\mu,(v\cdot \mathrm{DS}\cdot \mathrm{D})));\\
&i(v\cdot u,(\mathrm{S}\cdot u,(u^\mu,\mathrm{D}_\mu))),\ i(\mathrm{S}\cdot u,(v\cdot u,(\mathrm{D}^\mu,u_\mu))),\ i((v\cdot u,\mathrm{S}\cdot u),(\mathrm{D}^\mu,u_\mu)),\\
&i(\mathrm{D}_\mu,(u^\mu,(v\cdot u,\mathrm{S}\cdot u))),\ i(u_\mu,(\mathrm{D}^\mu,(v\cdot u,\mathrm{S}\cdot u)))
\end{aligned}
\tag{34}
$$

$i = 211, 212, 221, 253, ..260, 301$ of Table 2 form a set of linearly independent terms;

$$
\begin{aligned}
(c)\quad &i(\mathrm{D}_\mu,(\mathrm{S}\cdot \mathrm{D},(v\cdot \mathrm{D},u^\mu))),\ i(\mathrm{S}\cdot \mathrm{D},(\mathrm{D}_\mu,(v\cdot \mathrm{D},u^\mu))),\ i((\mathrm{S}\cdot \mathrm{D},\mathrm{D}_\mu),(v\cdot \mathrm{D},u^\mu)),\\
&i(u_\mu,(v\cdot \mathrm{D},(\mathrm{D}^\mu,\mathrm{S}\cdot \mathrm{D}))),\ i(v\cdot \mathrm{D},(u_\mu,(\mathrm{D}^\mu,\mathrm{S}\cdot \mathrm{D}))),
\end{aligned}
$$

$$
\begin{aligned}
&i(u_\mu, (\mathrm{S}\cdot u, (u^\mu, v\cdot \mathrm{D}))),\ \ i(\mathrm{S}\cdot u, (u_\mu, (u^\mu, v\cdot \mathrm{D}))),\ \ i((u_\mu, \mathrm{S}\cdot u), (u^\mu, v\cdot \mathrm{D}))),\\
&i(v\cdot \mathrm{D}, (u_\mu, (u^\mu, \mathrm{S}\cdot u)),\ \ i(u_\mu, (v\cdot \mathrm{D}, (u^\mu, \mathrm{S}\cdot u));\\
&i(u_\mu, (u^\mu, (\mathrm{S}\cdot u, v\cdot \mathrm{D}))),\ \ i(u^2, (\mathrm{S}\cdot u, v\cdot \mathrm{D})),\\
&i(v\cdot \mathrm{D}, (\mathrm{S}\cdot u, u^2)),\ \ i(\mathrm{S}\cdot u, (v\cdot \mathrm{D}, u^2)). \qquad (35)
\end{aligned}
$$

$i = 215, 216, 219, 269, .., 276, 303.$ of Table 2 form a set of linearly independent terms;

$$
\begin{aligned}
(d)\quad &i(\mathrm{D}_\mu, (\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}^\mu, v\cdot u))),\ \ i(\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}_\mu, (\mathrm{D}^\mu, v\cdot u))),\ \ i((\mathrm{D}_\mu, \mathrm{S}\cdot \mathrm{D}), (\mathrm{D}^\mu, v\cdot u))),\\
&i(v\cdot u, (\mathrm{D}_\mu, (\mathrm{D}^\mu, \mathrm{S}\cdot \mathrm{D})),\ \ i(\mathrm{D}_\mu, (v\cdot u, (\mathrm{D}^\mu, \mathrm{S}\cdot \mathrm{D}));\\
&i(u_\mu, (\mathrm{S}\cdot u, (v\cdot u, \mathrm{D}^\mu))),\ \ i(\mathrm{S}\cdot u, (u_\mu, (v\cdot u, \mathrm{D}^\mu))),\ \ i((\mathrm{S}\cdot u, u_\mu), (v\cdot u, \mathrm{D}^\mu)),\\
&i(\mathrm{D}_\mu, (v\cdot u, (u^\mu, \mathrm{S}\cdot u))),\ \ i(v\cdot u, (\mathrm{D}_\mu, (u^\mu, \mathrm{S}\cdot u)));\\
&i(\mathrm{D}_\mu, (\mathrm{D}^\mu, (\mathrm{S}\cdot \mathrm{D}, v\cdot u))),\ \ i(\mathrm{D}^2, (\mathrm{S}\cdot \mathrm{D}, v\cdot u)),\\
&i(v\cdot u, (\mathrm{S}\cdot \mathrm{D}, \mathrm{D}^2)),\ \ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot u, \mathrm{D}^2)) \qquad (36)
\end{aligned}
$$

$i = 217, 218, 220, 227, 302, 307, .., 312, 316$ of Table 2 form a set of linearly independent terms;

$$
\begin{aligned}
(e)\quad &i(v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, v\cdot u))),\ \ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, (v\cdot \mathrm{D}, v\cdot u))),\ \ i((v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}), (v\cdot \mathrm{D}, v\cdot u))),\\
&i(v\cdot u, (v\cdot \mathrm{D}, (v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D})),\ \ i(v\cdot \mathrm{D}, (v\cdot u, (v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}));\\
&i(v\cdot u, (\mathrm{S}\cdot u, (v\cdot u, v\cdot \mathrm{D}))),\ \ i(\mathrm{S}\cdot u, (v\cdot u, (v\cdot u, v\cdot \mathrm{D}))),\ \ i((\mathrm{S}\cdot u, v\cdot u), (v\cdot u, v\cdot \mathrm{D})),\\
&i(v\cdot u, (v\cdot \mathrm{D}, (v\cdot u, \mathrm{S}\cdot u)),\ \ i(v\cdot \mathrm{D}, (v\cdot u, (v\cdot u, \mathrm{S}\cdot u))\\
&i(v\cdot u, (\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D})^2)),\ \ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot u, (v\cdot \mathrm{D})^2)),\\
&i((v\cdot \mathrm{D})^2, (\mathrm{S}\cdot \mathrm{D}, v\cdot u)),\ \ i(v\cdot \mathrm{D}, (v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, v\cdot u)));\\
&i(v\cdot \mathrm{D}, \mathrm{S}\cdot u, (v\cdot u)^2)),\ \ i(\mathrm{S}\cdot u, (v\cdot \mathrm{D}, (v\cdot u)^2)),\\
&i((v\cdot u)^2, (\mathrm{S}\cdot u, v\cdot \mathrm{D})),\ \ i(v\cdot u, (v\cdot u, (\mathrm{S}\cdot u, v\cdot \mathrm{D})). \qquad (37)
\end{aligned}
$$

$i = 222, .., 225, 277, .., 284, 314$ of Table 2 form a set of linearly independent terms;

$$
\begin{aligned}
(f)\quad &i(v\cdot \mathrm{D}, (v\cdot \mathrm{D}, (v\cdot \mathrm{D}, \mathrm{S}\cdot u))),\ \ i((v\cdot \mathrm{D})^2, (v\cdot \mathrm{D}, \mathrm{S}\cdot u)),\\
&i[\mathrm{S}\cdot u, (v\cdot \mathrm{D})^3]_+,\ \ i(v\cdot \mathrm{D}, (\mathrm{S}\cdot u, (v\cdot \mathrm{D})^2)) \qquad (38)
\end{aligned}
$$

$i = 229, 230$ of Table 2 form a set of linearly independent terms. For $u \leftrightarrow \mathrm{D}$ in (38), $i = 305, 306$ of Table 2 form a set of linearly independent terms.

(2) Using $(18)'$ and (9), one sees that one has to consider the following set of terms together:

$$
\epsilon^{\mu\nu\rho\lambda}\Big((\mathrm{D}_\mu, (\mathrm{D}_\nu, (\mathrm{D}_\rho, u_\lambda))),\ \ ([\mathrm{D}_\mu, \mathrm{D}_\nu], (\mathrm{D}_\rho, u_\lambda))),
$$

$$
\begin{aligned}
&(u_\mu, (\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\lambda])),\ (\mathrm{D}_\mu, (u_\nu, [\mathrm{D}_\rho, \mathrm{D}_\lambda]));\\
&(u_\mu, (u_\nu, (u_\rho, \mathrm{D}_\lambda))),\ ([u_\mu, u_\nu], (u_\rho, \mathrm{D}_\lambda),\\
&(\mathrm{D}_\mu, (u_\nu, [u_\rho, u_\lambda])),\ (u_\mu, (\mathrm{D}_\nu, [u_\rho, u_\lambda]))\Big).
\end{aligned}
\tag{39}
$$

One can show that of the terms listed in (39), one can take $i = 231, 232, 233, 285, 286$ of Table 2 as a set of linearly independent terms.

A similar analysis is carried out to select $i = 234, 235, 236, 287, 288$; $237, 238, 239, 290, 291$ of Table 2.

Also, using $(18)'$ and (9), one can show that of the following set of terms:

$$
\begin{aligned}
&\epsilon^{\mu\nu\rho\lambda} v_\rho \Big((v\cdot \mathrm{D}, (\mathrm{D}_\mu, (\mathrm{D}_\nu, u_\lambda))),\ ((v\cdot \mathrm{D}, \mathrm{D}_\mu), (\mathrm{D}_\nu, u_\lambda))),\ (\mathrm{D}_\mu, (v\cdot \mathrm{D}, (\mathrm{D}_\nu, u_\lambda))),\\
&(u_\mu, (\mathrm{D}_\nu, (v\cdot \mathrm{D}, \mathrm{D}_\lambda))),\ (\mathrm{D}_\mu, (u_\nu, (v\cdot \mathrm{D}, \mathrm{D}_\lambda)));\\
&(v\cdot u, (u_\mu, (u_\nu, \mathrm{D}_\lambda))),\ (u_\mu, (v\cdot u, (u_\nu, \mathrm{D}_\lambda))),\ ((u_\mu, v\cdot u), (u_\nu, \mathrm{D}_\lambda)),\\
&(\mathrm{D}_\mu, (u_\nu, (v\cdot u, u_\lambda))),\ (u_\mu, (\mathrm{D}_\nu, (v\cdot u, u_\lambda)))\Big),
\end{aligned}
\tag{40}
$$

one need consider 13 independent terms, say, $i = 240, .., 243, 292, .., 300$ of Table 2.

<u>$p \neq 0$ or $q \neq 0$ in (4)$\equiv (A, (B, C))$</u>

<u>(v) of $(7)'$</u>

Using (9) and $(28)'$, one sees that one will have to consider the following set of terms together:

$$
\begin{aligned}
&(\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, \chi_-)),\ (v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, \chi_-)),\ ((v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}), \chi_-);\\
&(v\cdot u, (\mathrm{S}\cdot u, \chi_-)),\ (\mathrm{S}\cdot u, (v\cdot u, \chi_-)),\ ((\mathrm{S}\cdot u, v\cdot u), \chi_-).
\end{aligned}
\tag{41}
$$

Further noting that u_μ and χ_- are isovectors, we see that

$$
[v\cdot u, [\mathrm{S}\cdot u, \chi_-]_+] = [\mathrm{S}\cdot u, [v\cdot u, \chi_-]_+] = [[v\cdot u, \mathrm{S}\cdot u]_+, \chi_-] = 0. \tag{42}
$$

Applying (9), $(28)'$ and (42) to (41), we see that we get two linearly independent terms, say, $[\mathrm{S}\cdot \mathrm{D}, [v\cdot \mathrm{D}, \chi_-]]_+,\ i[\mathrm{S}\cdot u, [v\cdot u, \chi_-]]_+$.

<u>(vi) of $(7)'$</u>

Using $(31)'$, one sees that one will have to consider the following set of terms together:

$$
i(\mathrm{S}\cdot \mathrm{D}, (v\cdot u, \chi_+)),\ i(v\cdot u, (\mathrm{S}\cdot \mathrm{D}, \chi_+)),\ i((\mathrm{S}\cdot \mathrm{D}, v\cdot u), \chi_+); v \leftrightarrow \mathrm{S}. \tag{43}
$$

Further, noting that $\mathrm{D}_\mu \equiv$isoscalar$(\equiv \partial_\mu)$ +isovector$(\equiv \Gamma_\mu)$, u_ν is an isovector and χ_+ is an isoscalar, we see

$$[[\mathrm{S}\cdot \mathrm{D}, v\cdot u], \chi_+] = [[v\cdot \mathrm{D}, \mathrm{S}\cdot u], \chi_+] = [v\cdot u, [\mathrm{S}\cdot \mathrm{D}, \chi_+]] = [\mathrm{S}\cdot u, [v\cdot \mathrm{D}, \chi_+]] = 0. \tag{44}$$

Applying (9), $(31)'$ and (44) to (43), we see that we get one linearly independent term, say, $[[\mathrm{S}\cdot \mathrm{D}, v\cdot u]_+, \chi_+]_+$,; similarly for $v \leftrightarrow \mathrm{S}$.

Note that because of parity constraints and the algebra of the S_μs (See [1]), there are no Levi Civita-dependent (2,0,0,1)-, (0,2,0,1)- and (1,1,1,0)-type terms.

4 The Lists of Independent Terms in $\mathcal{L}_{\mathrm{HBChPT}}$ (off-shell nucleons)

In this section, using (6), and the algebraic reductions of Section 3, we list all possible $\mathcal{A}$-type terms of $\mathrm{O}(q^4, \phi^{2n})$, and $\mathrm{O}(q^4, \phi^{2n+1})$ in Tables 1 and 2, that are allowed by (6) and have not been eliminated in Section 3. As noted in Section 2 (and [1]), for off-shell nucleons, $\gamma^0\mathcal{B}^\dagger\gamma^0\mathcal{C}^{-1}\mathcal{B} \in \mathcal{A}$. Hence, it is sufficient to list only $\mathcal{A}$-type terms (for off-shell nucleons).

Let us summarize the basis of construction of list of linearly independent terms using (6) and section 3. Using the notation of [11], one groups terms allowed by (6) as in (7). Then, following Section 3, we write down all possible identities by considering different groups of terms together (e.g. (*iii*) and (*iv*) of (7)). Sometimes, one has to consider together several types of terms but belonging to the same types of groups. There are two different cases to be considered: (a) the different term types involve different permutations of the same building blocks, and (b) some of the building blocks of the different term types are different. We will illustrate both cases by considering an example each.

(a) For example, one has to consider the triplet of terms [3]

$$\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda(\mathrm{D}_\mu, ((\mathrm{D}_\nu, u_\lambda), \mathrm{S}\cdot u)) \in (iii) \text{ of } (7),\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda(\mathrm{S}\cdot u, ((\mathrm{D}_\nu, u_\lambda), \mathrm{D}_\mu)) \in (iii)(\text{“}A \leftrightarrow B\text{”}) \text{ of } (7),\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda((\mathrm{D}_\mu, \mathrm{S}\cdot u), ((\mathrm{D}_\nu, u_\lambda)), \in (iv) \text{ of } (7),
\end{aligned} \tag{45}$$

in conjunction with

$$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda(\mathrm{D}_\mu, ((\mathrm{D}_\nu, \mathrm{S}\cdot u), u_\lambda)) \in (iii) \text{ of } (7),$$

[3] The reason for considering the terms as triplets is because (18) shows there are linear relations involving them.

$$
\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda(u_\lambda, ((\mathrm{D}_\nu, \mathrm{S}\cdot u), \mathrm{D}_\mu)) \in (iii)(\text{“}A \leftrightarrow B\text{”}) \text{ of } (7),\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda((\mathrm{D}_\mu, u_\lambda), (\mathrm{D}_\nu, \mathrm{S}\cdot u)) \in (iv) \text{ of}(7).
\end{aligned}
\tag{46}
$$

[Note that $i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda(\mathrm{D}_\mu, ((\mathrm{D}_\nu, u_\lambda), \mathrm{S}\cdot u))$ and $i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda(\mathrm{D}_\mu, ((\mathrm{D}_\nu, \mathrm{S}\cdot u), u_\lambda))$ both $\in (iii)$ of (7), and so forth.] The reason for doing so is, as we notice in the example, $i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda((\mathrm{D}_\mu, u_\lambda), (\mathrm{D}_\nu, \mathrm{S}\cdot u)) \in (iv)$ of (7), is common to both the triplets.

(b) For example, one needs to consider the following triplet of terms [4] :

$$
(v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, \chi_-)),\ (\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, \chi_-)),\ ((v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}, \chi_-)
\tag{47}
$$

in conjunction with

$$
[v\cdot u, [\mathrm{S}\cdot u, \chi_-]]_+,\ [\mathrm{S}\cdot u, [v\cdot u, \chi_-]]_+,\ [[v\cdot u, \mathrm{S}\cdot u], \chi_-]_+.^5
\tag{48}
$$

This is so because of (9).

Using the algebraic identities of Section 3, if we end up with m independent identities in $n(>m)$ terms, then we can take $(n-m)$ linearly independent terms. So, for the above examples, (a) $m = 6+6 = 12$ and $n = 12+8 = 20$, implying that one can take eight linearly independent terms; (b) $m = 7$ and $n = 9$ implying that one can take two linearly independent terms. Care has to be taken in the choice of those $(n-m)$ terms, namely, no subset of these terms should satisfy any relations. Let us first consider the example in (a). A valid choice is the set of eight terms in Table 1: $i = 99, .., 106$. The following set of eight terms, however, *are not* linearly independent: $iv_\rho\epsilon^{\mu\nu\rho\lambda}\Big([\mathrm{D}_\mu, [[\mathrm{D}_\nu, u_\lambda]_+, \mathrm{S}\cdot u]_+]; [\mathrm{S}\cdot u, [[\mathrm{D}_\nu, u_\lambda]_+, \mathrm{D}_\mu]_+];\ [[\mathrm{D}_\mu, \mathrm{S}\cdot u]_+, [\mathrm{D}_\nu, u_\lambda]_+];\ [\mathrm{D}_\mu, [[\mathrm{D}_\nu, u_\lambda], \mathrm{S}\cdot u]]\Big)$; $i = 101, .., 104$(of Table 1). The reason is that from the first identity in (18), one sees that the first three terms are not linearly independent. Note that the allowed set of eight terms is *not a unique choice*. An equivalent choice would be $iv_\rho\epsilon^{\mu\nu\rho\lambda}[[\mathrm{D}_\mu, \mathrm{S}\cdot u]_+, [\mathrm{D}_\nu, u_\lambda]_+]$; $i = 99, .., 101, 103, .., 106$ of Table 1. Now, coming to the example considered in (b). Two valid equivalent choices are: $[\mathrm{S}\cdot\mathrm{D}, [v\cdot\mathrm{D}, \chi_-]]_+$, $[\mathrm{S}\cdot u, [v\cdot u, \chi_-]]_+$ or $[\mathrm{S}\cdot\mathrm{D}, [v\cdot\mathrm{D}, \chi_-]_+]$, $[[\mathrm{S}\cdot u, v\cdot u], \chi_-]_+$; and, e.g., $[\mathrm{S}\cdot u, [v\cdot u, \chi_-]]_+$, $[v\cdot u, [\mathrm{S}\cdot u, \chi_-]]_+$ are not linearly independent.

[4] The reason for considering the terms as triplets is because (28) shows that there are linear relations between them.

[5] Note that because of (42), one need not consider $[v\cdot u, [\mathrm{S}\cdot u, \chi_-]_+]$, $[\mathrm{S}\cdot u, [v\cdot u, \chi_-]_+]$, $[[v\cdot u, \mathrm{S}\cdot u]_+, \chi_-]$.

Even though the phase rule (6) and linear independence of terms are sufficient for listing terms in the O(q^4) HBChPT Lagrangian for off-shell nucleons, however, if for a given choice of terms and group of terms in (7), we find similar group of terms in [7], then while listing the $(n-m)$ terms, preference is given to including terms that also figure in Table 1 of [7]. The reason for doing the same is that this allows for an easy identification of the finite terms, given that the divergent (counter) terms have been worked out in [7].

In tables 1 and 2, the allowed 4-tuples (m, n, p, q) are listed along with the corresponding terms. The main aim is to find the number of finite O(q^4) terms, given that the UV divergent terms have already been worked out in [7]. For this purpose, the terms in tables 1 and 2 are labeled as F denoting the finite terms and D denoting the divergent terms. For the purpose of comparison with [7], we have also indicated which terms in table 1 of [7] (setting the external fields to zero and assuming isospin symmetry) the D-type terms correspond to. The LECs of O(q^4) terms in [7] are denoted by $d_i, i = 1$ to 199. Further, the $i = 188$ term in Table 1 of [7] should have S_ρ instead of v_ρ.

4.1 O(q^4, ϕ^{2n}) Terms

These terms are listed in Table 1. One gets a total of 207 O(q^4, ϕ^{2n}) terms, the LECs of two of which, as will be shown in Section 5, are fixed relative to those of lower order terms. The last column of Table 1 will be explained in Section 7.

4.2 O(q^4, ϕ^{2n+1}) Terms

These terms are listed in Table 2. One gets a total of 113 O(q^4, ϕ^{2n+1}) terms. The last column of Table 2 will ll be explained in Section 7.

Overall, one gets 230 finite and 90 divergent (counter) terms at O(q^4).

5 Further Reduction in LECs due to Reparameterization Invariance

In this section, we discuss those O(q^4) terms whose low energy coupling constants (LECs) are fixed relative to O($q^{1,2,3}$) terms.

We first show that for off-shell nucleons, all such O(q^4) $A-$type terms belong to a certain class of (2,2,0,0)-type of terms, and the O(q^2, ϕ^{2n}) term is

$\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$, and the $\mathrm{O}(q^3,\phi^{2n})$ term is $i\bar{\mathrm{H}}u_\mu v\cdot u\mathrm{D}^\mu\mathrm{H}$+h.c. We then show that by using the formulation of Luke and Manohar [12] for what is called reparameterization invariance (RI), one arrives at the same conclusion, implying that the above reduction in the number of independent LEC's is equivalent to reparameterization invariance. Then, for on-shell nucleons, we see that even though RI poses no constraints on the $A-$type terms, but as its consequence, the LECs of quite a few terms arising from $(\gamma^0 B^\dagger\gamma^0 C^{-1}B)^{(4)}$, are fixed relative to the LECs of lower order terms.

5.1 Off-Shell Nucleons

For this subsection, even though not explicitly written everywhere, but a nonrelativisitc term will imply an $A-$type HBChPT term. This is because for off-shell nucleons, $\gamma^0 B^\dagger\gamma^0 C^{-1}B\in A$.

In general, a nonrelativistic term can be written as:

$$\left(\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\right)\bar{\mathrm{H}}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}\mathcal{A}_{\nu_j}(\mathcal{V}_\alpha\mathcal{V}^\alpha)^{m_1}(\mathcal{A}_\beta\mathcal{A}^\beta)^{m_2}(\mathcal{V}_\mu\mathcal{A}^\mu)^{m_3}(\chi_+)^p(\chi_-)^q\mathrm{H}, \tag{49}$$

and $\mathcal{V}_{\mu_i}\equiv v_{\mu_i}$ or D_{μ_i} and $\mathcal{A}_{\nu_j}\equiv u_{\nu_j}$ or $\mathrm{S}_{\nu_\mathrm{j}}$ and only tensor contractions are indicated in (49). The upper component H and $\bar{\mathrm{H}}$ will be dropped in the equations and tables after equation (52).

Using

$$\bar{\mathrm{H}}\mathrm{D}^2\mathrm{H}\equiv\bar{\psi}\mathrm{P}_+\left(-(\mathrm{i}\not{\mathrm{D}}-\mathrm{m})^2+\frac{\mathrm{i}}{8}\sigma^{\mu\nu}[\mathrm{u}_\mu,\mathrm{u}_\nu]\right)\mathrm{P}_+\psi, \tag{50}$$

($P_+\equiv\frac{1}{2}(1+\not{v})$) and

$$-2i\mathrm{m}\bar{\mathrm{H}}v\cdot\mathrm{DH}=\bar{\psi}P_+(\mathrm{D}^2+\mathrm{m}^2)P_+\psi-\bar{\mathrm{H}}\mathrm{D}^2\mathrm{H} \tag{51}$$

one gets:

$$-2i\mathrm{m}\bar{\mathrm{H}}v\cdot\mathrm{DH}\equiv\bar{\psi}P_+\left((\mathrm{D}^2+\mathrm{m}^2)-\left(-(i\not{\mathrm{D}}-\mathrm{m})^2+\frac{i}{8}\sigma^{\mu\nu}[u_\mu,u_\nu]\right)\right)P_+\psi\ . \tag{52}$$

Using (50) and (52) one can construct the Tables 3 and 5. In these tables, by considering all possible terms up to $\mathrm{O}(q^4)$, it is shown that one needs to consider only one kind of (2,2,0,0) term in the Levi-Civita(L.C)-independent category (Table 4), whose LECs are going to be fixed relative to the LECs

of lower order terms ($\mathrm{O}(q^2, q^3)$). In Tables 3 and 5 $(\chi_+)^p(\chi_-)^q$ have been dropped because they are the same in HBChPT and BChPT.

(a) Table 3 (L.C.-independent terms):

We consider $(\mathcal{V}_\alpha\mathcal{V}^\alpha)^{m_1}$, $(\mathcal{A}_\beta\mathcal{A}^\beta)^{m_2}$ and $(\mathcal{V}_\mu\mathcal{A}^\mu)^{m_3}$ separately and see if they and their relativistic counterparts are of the same chiral order. This is done in Table 3. To understand Table 3 better, let us consider the example belonging to the $(v\cdot\mathrm{D})^{l_1}(\mathrm{D}_\nu\mathrm{D}^\nu)^{\mathrm{l}_2}$-type terms. A term in Table 1 that belongs to this category is $\mathrm{D}_\mu v\cdot\mathrm{D}\mathrm{D}^\mu v\cdot\mathrm{D}) + \mathrm{h.c.}$. What Table 3 says is that it is possible to find a relativistic counterpart of this term which is of the same chiral order $\equiv 0(q^4)$. One can explicitly check that $-\frac{1}{4\mathrm{m}^2}\mathrm{D}_\mu\{\}\mathrm{D}^\mu - \frac{1}{4}\{\}^2 + \frac{1}{4\mathrm{m}^2}\{\}^3 + \mathrm{h.c.}$, on nonrelativistic reduction gives $\mathrm{D}_\mu v\cdot\mathrm{D}\mathrm{D}^\mu v\cdot\mathrm{D} + \mathrm{h.c.}$; $\{\}\equiv(\mathrm{D}^2+\mathrm{m}^2) - \left(-(i\not{\mathrm{D}}-\mathrm{m})^2 + \frac{i}{8}\sigma^{\mu\nu}[u_\mu, u_\nu]\right)$.

From inspection of Table 3, it becomes clear that for Levi-Civita-independent terms, except for $(u_\mu\mathrm{D}^\mu)^{\mathrm{j}_2}\in(\mathcal{V}_\mu\mathcal{A}^\mu)^{m_3}$, the nonrelativistic terms are of the same chiral order as their relativistic counterparts. This exception is because the $\frac{1}{\mathrm{m}}$-reduction of $\bar\psi(u_\mu\mathrm{D}^\mu)^{\mathrm{j}_2}\psi$ gives, in addition to the expected term $\bar{\mathrm{H}}(u_\mu\mathrm{D}^\mu)^{\mathrm{j}_2}\mathrm{H}$, also lower order terms. For example, for $j_2 = 2$, one gets $i\bar{\mathrm{H}}v\cdot uu_\mu\mathrm{D}^\mu\mathrm{H}$ and $\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$, which are of $\mathrm{O}(q^3)$ and $\mathrm{O}(q^2)$. For $j_2\geq 2$, it becomes impossible to find a linear combination of relativistic counterparts that will exactly cancel the lower order terms because there will be terms of at least two different lower orders (which is the case for $j_2 = 2$) to be taken care of. For $j_2 = 1$, it is possible to construct a linear combination of relativistic counterparts that gives only the non-relativistic terms one is interested in, canceling the lower order terms, e.g. for $\bar{\mathrm{H}}iu^\mu v\cdot u\mathrm{D}_\mu\mathrm{H} + \mathrm{h.c.}$ ($\equiv i = 21$ $\mathrm{O}(q^3, \phi^{2n})$ term in [1]). So, there is no reduction in $\mathrm{O}(q^3)$ terms.

Up to $\mathrm{O}(q^4)$, it is sufficient to consider only $\bar{\mathrm{H}}(u_\mu\mathrm{D}^\mu)^2\mathrm{H}$. Its relativistic counterpart $\bar\psi(u_\mu\mathrm{D}^\mu)^2\psi$, on $\frac{1}{\mathrm{m}}$-reduction gives $\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$ as the $\mathrm{O}(q^2)$ and $-i\mathrm{m}\bar{\mathrm{H}}[v\cdot u(u_\mu\mathrm{D}^\mu)+(u_\mu\mathrm{D}^\mu)\mathrm{v}\cdot\mathrm{u}]\mathrm{H}$ as the $\mathrm{O}(q^3)$ non-relativistic terms. One can modify the relativistic counterpart in two ways to eliminate one of the two terms of different lower orders. One can eliminate the $\mathrm{O}(q^3)$ non-relativistic term, by considering $\bar\psi[(u_\mu\mathrm{D}^\mu)^2 + i\mathrm{m}\not{u}(\mathrm{u}_\mu\mathrm{D}^\mu) + i\mathrm{m}(u_\mu\mathrm{D}^\mu)\not{u}]\psi$. However, this way, one also gets the $\mathrm{O}(q^2)$ term $\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$, implying that the relativistic counterpart is of $\mathrm{O}(q^2)$. Alternatively, one can eliminate the $\mathrm{O}(q^2)$ non-relativistic term by considering $\bar\psi[(u_\mu\mathrm{D}^\mu)^2 + i\mathrm{m}\not{u}(u_\mu\mathrm{D}^\mu)]\psi$. This way one also gets the $\mathrm{O}(q^3)$ term $i\bar{\mathrm{H}}u\cdot\mathrm{D}v\cdot u\mathrm{H}$, implying that the relativistic counterpart is of $\mathrm{O}(q^3)$. Hence, the LEC's of $\bar{\mathrm{H}}(u^\mu\mathrm{D}^\mu)^2\mathrm{H}$ is fixed relative to $\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$ and $i\bar{\mathrm{H}}u_\mu v\cdot u\mathrm{D}^\mu$.

Using the algebraic reductions of Section 3, one sees that there are two

$\mathrm{O}(q^4,\phi^{2n})$ (L.C.-independent) terms whose LECs are fixed relative to the $\mathrm{O}(q^2)$ term $\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$ and the $\mathrm{O}(q^3)$ term $i\bar{\mathrm{H}}u_\mu v\cdot u\mathrm{D}^\mu\mathrm{H}$ + h.c.- given in Table 4.

We will now show (in Table 5) that up to $\mathrm{O}(q^4)$, one gets no such reduction in the number of independent $\mathrm{O}(q^4)$ LECs for L.C.-dependent terms.

(b) Table 5 (L.C.-dependent terms):

Using the conclusion arrived upon from Table 3, it is sufficient to consider:

$$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}\mathcal{A}_{\nu_j}\times\Big(1,\ (\mathcal{A}_\mu\mathcal{V}^\mu)^{m_3}\Big), \tag{53}$$

i.e. one need not consider $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}\mathcal{A}_{\nu_j}(\mathcal{V}_\alpha\mathcal{V}^\alpha)^{m_1}(\mathcal{A}_\beta\mathcal{A}^\beta)^{m_2}$. Using the algebra of S_μs and the antisymmetry of $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}$, it is sufficient to consider L.C.-dependent terms with one or no S_μ and one or no v_μ. In Table 5, one sees whether (53) up to $\mathrm{O}(q^4)$ and their relativistic counterparts are of of the same chiral order.

To understand Table 5 better, let us consider the example of $\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_{\nu_l}\prod_{i=1}\mathrm{D}_{\mu_i}\prod_{j\neq l}u_{\nu_j}(v\cdot u)$ in Table 5. A term in Table 1 that belongs to this class of terms is $i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,v\cdot u]]_+$. What Table 5 tells is that it is possible to construct the relativistic counterpart of the same chiral order (i.e $\mathrm{O}(q^4)$). One can explicitly check that $\frac{\epsilon^{\mu\nu\rho\lambda}}{\mathrm{m}}\bar{\psi}\gamma^5\gamma_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,[\mathrm{D}_\kappa,u^\kappa]_+]]_+\psi$ $+4i\mathrm{m}\bar{\psi}\sigma^{\nu\lambda}[[\mathrm{D}_\nu,[\mathrm{D}_\kappa,u^\kappa]_+],u_\lambda]_+\psi\equiv O(q^4)$ after nonrelativistic reduction gives $i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,v\cdot u]]_+\equiv O(q^4)$, as one of its $\mathrm{O}(q^4)$ terms. [6] From Table 5, one sees that for Levi-Civita-dependent terms, the nonrelativistic terms are of the same chiral order as their relativistic counterparts.

The LECs of the $\mathrm{O}(q^4,\phi^{2n})$ terms of Table 4, are also fixed relative to the LEC's of lower order terms because of the "cross terms" $\gamma^0\mathcal{B}^\dagger\gamma^0\mathcal{C}^{-1}\mathcal{B}$ in (5). One can show that the $B^{(3)}$ obtained from the nonrelativistic reduction of $i\bar{\psi}[[\mathrm{D}_\mu,u^\mu]_+,\not{u}]_+\psi,\ \bar{\psi}[[\mathrm{D}_\mu,\not{u}]_+,u^\mu]_+\psi, i\bar{\psi}[\mathrm{D}^\mu,[\not{u},u_\mu]_+]_+\psi$ together with the $B^{(1)}$ obtained from the nonrelativistic reduction of $i\bar{\psi}\not{D}\psi$, using $\gamma^0\mathcal{B}^\dagger\gamma^0\mathcal{C}^{-1}\mathcal{B}$ give $\bar{\mathrm{H}}[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\nu]_+,u^\mu]_+]_+\mathrm{H}$, $\bar{\mathrm{H}}[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\mu]_+,u^\nu]_+]_+\mathrm{H}$ and $\bar{\mathrm{H}}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u^\mu,u^\nu]_+]_+]_+\mathrm{H}$. This implies that the LEC of $\bar{\mathrm{H}}[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\nu]_+,u^\mu]_+]_+\mathrm{H}$, $\bar{\mathrm{H}}[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\mu]_+,u^\nu]_+]_+\mathrm{H}$ and $\bar{\mathrm{H}}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u^\mu,u^\nu]_+]_+]_+\mathrm{H}$ will also be fixed relative to the Dirac term.

[6] Strictly speaking, one will also get contributions from the "cross terms" $\gamma^0\mathcal{B}^\dagger\gamma\mathcal{C}^{-1}\mathcal{B}$: $[[[v\cdot\mathrm{D},v\cdot u],\mathrm{S}\cdot u]_+,v\cdot\mathrm{D}]$, $[[[\mathrm{S}\cdot\mathrm{D},v\cdot u],v\cdot u]_+,v\cdot\mathrm{D}]$, $[[[v\cdot\mathrm{D},v\cdot u],u^\mu]_+,u_\mu]_+$, $[[[v\cdot\mathrm{D},v\cdot u],v\cdot u]_+,v\cdot u]_+$, $\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[[[v\cdot\mathrm{D},v\cdot u],u_\mu]_+,u_\nu]_+$, $i[[[\mathrm{D}_\mu,v\cdot u],v\cdot u]_+,u^\mu]_+$, $i[[[v\cdot\mathrm{D},v\cdot u],v\cdot u]_+,v\cdot u]_+$, $\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[[[\mathrm{D}_\mu,v\cdot u],v\cdot u]_+,u_\nu]_+$. One can show that all these terms can be obtained independently from the nonrelativistic reduction of other relativistic terms.

So, the conclusion one arrives at from the discussion so far, is that for O(q^4) terms, it is only a certain class of (2,2,0,0) terms whose LECs are fixed relative to the O(q^m, ϕ^{2n}), $m = 1, 2, 3$ terms. It should be noted that all three terms of Table 4 are such that the derivatives act on the baryon field, in addition to acting on the meson field.

Now we show the connection between the above and reparameterization invariance (RI).

Using the formalism of Luke and Manohar [12], $\mathcal{L}_{\rm HBChPT}$ can be written in terms of manifestly reparameterization invariant nucleon field $\mathcal{H}_v$:

$$\mathcal{H}_v \equiv \Lambda(\frac{p}{\rm m}, v){\rm H}_v = (1 + i\frac{\not{D}}{2{\rm m}}){\rm H}_v + \mathcal{O}(\frac{1}{{\rm m}^2}), \tag{54}$$

where $\Lambda(\frac{p}{\rm m}, v)$ represents the Lorentz-boost matrix relating ${\rm H}_{\frac{p}{\rm m}}$ to ${\rm H}_v$ (${\rm H}_v$ has been hitherto denoted by H; the label v was assumed), (p is the total 4-momentum of the nucleon), and the unimodular reparameterization invariant velocity operator "$\mathcal{U}_\mu$" $\equiv \frac{\mathcal{V}_\mu}{|\mathcal{V}|}$, where $\mathcal{V}_\mu \equiv v_\mu + i\frac{{\rm D}_\mu}{\rm m}$, and $|\mathcal{V}| \equiv \sqrt{\mathcal{V}^2}$ $= \sqrt{1 - 2i\frac{v\cdot {\rm D}}{\rm m} - \frac{{\rm D}^2}{{\rm m}^2}}$.

$$\mathcal{U}_\mu = \mathcal{V}_\mu + iv_\mu\frac{v\cdot {\rm D}}{\rm m} + O(\frac{1}{{\rm m}^2}). \tag{55}$$

We will consider one example from Tables 4. The first term in Table 4 can be written as:

$$\bar{\mathcal{H}}_v[\mathcal{U}_\mu, [u_\nu, [\mathcal{U}^\nu, u^\mu]_+]_+]_+\mathcal{H}_v. \tag{56}$$

While using (54) (and (55)) to rewrite (56) in terms of H, ${\rm D}_\mu$ (and u_ν), one can replace $\mathcal{H}_v$ by H because the $i\not{D}$ occurs as $v\cdot{\rm D}$ or ${\rm D}^2$ or $\sigma^{\mu\nu}[u_\mu, u_\nu]$ (using the curvature relation), which can be obtained from other reparameterization invariant terms containing $\mathcal{U}^2 - 1$ and $\sigma^{\mu\nu}[u_\mu, u_\nu]$. [7] After doing so, one gets:

$$-\frac{1}{{\rm m}^2}\bar{\rm H}\Big([{\rm D}_\mu, [u_\nu, [{\rm D}^\nu, u^\mu]_+]_+]_+ - 2i{\rm m}[{\rm D}^\nu, [v\cdot u, u_\nu]_+]_+ \\ -2i{\rm m}[u_\mu, [{\rm D}^\mu, v\cdot u]_+]_+ - 8{\rm m}^2(v\cdot u)^2\Big){\rm H}. \tag{57}$$

[7]Strictly speaking, this is not a complete argument for dropping the $i\not{D}$ in $\mathcal{H}_v$ because $\mathcal{U}^2 - 1$ generates both $iv\cdot{\rm D}$ and ${\rm D}^2$. For off-shell nucleons, LM's formalism does not show how to obtain $iv\cdot{\rm D}$ and ${\rm D}^2$ independently from different manifestly reparameterization invariant terms. But using (50) and (52), for off-shell nucleons, both $iv\cdot{\rm D}$ and ${\rm D}^2$ can be obtained from independent HBChPT terms (written in terms of H).

Now, $i[\mathrm{D}^\nu,[v\cdot u,u_\nu]_+]_+ \equiv iv\cdot uu\cdot\mathrm{D}+iu_\mu v\cdot u\mathrm{D}^\mu + \text{h.c.} \equiv 2iu_\mu v\cdot u\mathrm{D}^\mu + \text{h.c.}$ ($\equiv i=21$ term in [1])$+i[\mathrm{D}^\nu,[u_\nu,v\cdot u]]$. One need not consider $i[\mathrm{D}_\mu,[u^\mu,v\cdot u]]$ as it can be obtained independently. Similarly, $i\bar{\mathrm{H}}[u_\mu,[\mathrm{D}^\mu,v\cdot u]_+]_+\mathrm{H} = 2i\bar{\mathrm{H}}u_\mu v\cdot u\mathrm{D}^\mu\mathrm{H}+\text{h.c.}+ i\bar{\mathrm{H}}[u_\mu,[\mathrm{D}^\mu,v\cdot u]]\mathrm{H}$; $i\bar{\mathrm{H}}[u_\mu,[\mathrm{D}^\mu,v\cdot u]]\mathrm{H}$ can be obtained independently. So, one sees that the LEC of $\bar{\mathrm{H}}[\mathrm{D}_\mu,[u_\nu,[\mathrm{D}^\nu,u^\mu]_+]_+]_+\mathrm{H}$ is fixed relative to the LECs of $\bar{\mathrm{H}}(v\cdot u)^2\mathrm{H}$ and $i\bar{\mathrm{H}}u_\mu v\cdot u\mathrm{D}^\mu\mathrm{H}$. One can similarly show the same for the other terms in Table 4.

5.2 On-Shell Nucleons

For on-shell nucleons, by the application of (66), all three terms in Table 4 get eliminated as A-type terms. However, the first as well as 43 other terms, arising from $(\gamma^0 B^\dagger\gamma^0 C^{-1}B)^{(4)}$, have their LECs fixed relative to $\mathrm{O}(q^{1,2})$ terms. The following is the list of the 44 terms:

$$
\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{D}_\lambda u_\mu u_\nu\mathrm{S}\cdot\mathrm{D}+\text{h.c.};\; i\mathrm{S}\cdot u[u^\mu,v\cdot u]\mathrm{D}_\mu+\text{h.c.};\\
&iu^\mu[\mathrm{S}\cdot u,v\cdot u]\mathrm{D}_\mu+\text{h.c.};\; iu^\mu[u_\mu,v\cdot u]\mathrm{S}\cdot\mathrm{D}+\text{h.c.};\\
&i\mathrm{D}^\mu\mathrm{S}\cdot uu_\mu v\cdot u+\text{h.c.};\; i\mathrm{D}^\mu u_\mu\mathrm{S}\cdot uv\cdot u+\text{h.c.};\\
&i[\mathrm{S}\cdot\mathrm{D},(v\cdot u)^3]_+;\; i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\mu(v\cdot u)^2\mathrm{D}_\nu;\\
&\mathrm{D}_\mu u^2\mathrm{D}^\mu;\; i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\mu u^2\mathrm{D}_\nu;\\
&\mathrm{D}_\mu\chi_+\mathrm{D}^\mu;\; i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\mu\chi_+\mathrm{D}_\nu;\\
&iv\cdot u\chi_+\mathrm{S}\cdot\mathrm{D}+\text{h.c.};\; i[v\cdot u,u_\mu]u^\mu\mathrm{S}\cdot\mathrm{D}+\text{h.c.};\\
&i[v\cdot u,\mathrm{S}\cdot u]u_\mu\mathrm{D}^\mu+\text{h.c.};\; i\chi_-u^\mu\mathrm{D}_\mu+\text{h.c.};\\
&\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\chi_-u_\mu\mathrm{D}_\nu+\text{h.c.};\\
&i[u_\mu,v\cdot u]\mathrm{S}\cdot u\mathrm{D}^\mu+\text{h.c.};\; u_\mu\mathrm{D}_\nu u^\mu\mathrm{D}^\nu+\text{h.c.};\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\kappa u_\mu\mathrm{D}^\kappa u_\nu+\text{h.c.};\; i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}\cdot u\mathrm{D}_\mu u_\lambda\mathrm{D}_\nu+\text{h.c.};\\
&i\mathrm{D}_\mu u^\mu v\cdot u\mathrm{S}\cdot u+\text{h.c.};\\
&i\mathrm{S}\cdot\mathrm{D}u^2v\cdot u+\text{h.c.};\; \mathrm{D}^\mu u^\nu u_\mu\mathrm{D}_\nu+\text{h.c.};\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}^\kappa[u_\mu,u_\kappa]\mathrm{D}_\nu;\; i\mathrm{D}^\mu[v\cdot u,u_\mu]_+\mathrm{S}\cdot u+\text{h.c.};\\
&i\mathrm{D}_\mu[\mathrm{S}\cdot u,u_\mu]v\cdot u+\text{h.c.};\; i[[u^2,v\cdot u]_+,\mathrm{S}\cdot\mathrm{D}]_+;\; i[u^\mu v\cdot uu_\mu,\mathrm{S}\cdot\mathrm{D}]_+;\\
&i[\mathrm{D}^\mu,[v\cdot u,[u^\mu,\mathrm{S}\cdot u]]_+];\; i[\mathrm{S}\cdot\mathrm{D},[v\cdot u,\chi_+]_+]_+;\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\nu,[v\cdot u,[\mathrm{D}_\mu,v\cdot u]]];\; [\mathrm{D}_\mu,[u_\nu,[\mathrm{D}_\mu,u^\nu]]]_+;\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\kappa,[\mathrm{D}_\mu,u^\kappa]]];\; [\mathrm{D}^\nu,[u^\mu,[\mathrm{D}_\mu,u_\nu]_+]_+]_+;\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\nu,[u^\kappa,[u_\mu,\mathrm{D}_\kappa]_+]_+];\; [\mathrm{D}^\nu,[u^\mu,[u_\nu,\mathrm{D}_\mu]]]_+;\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\nu,[u^\kappa,[u_\mu,\mathrm{D}_\kappa]]];\; i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\nu,[v\cdot u,[\mathrm{D}_\mu,v\cdot u]]_+]_+;
\end{aligned}
$$

$$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\mu,[v\cdot\mathrm{D},v\cdot u]]_+]_+;\ i[\mathrm{D}_\mu,[\chi_-,u^\mu]]_+;$$
$$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[\chi_-,u_\mu]];\ i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[v\cdot u,[v\cdot\mathrm{D},u_\mu]]];$$
$$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\mu,[v\cdot\mathrm{D},v\cdot u]]]. \tag{58}$$

Their coefficients can be determined from (74), (75) and (77) of Section 7.

Hence, in conclusion, for off-shell nucleons, one gets 207 $\mathrm{O}(q^4,\phi^{2n})$ and 113 $\mathrm{O}(q^4,\phi^{2n+1})$ terms. Of these, reparameterization/Lorentz invariance fixes the LECs of of three terms of the L.C.-dependent (2,2,0,0) category given in Table 4; no such reduction in the number of independent LECs is obtained in the L.C.-dependent terms up to $\mathrm{O}(q^4)$. As a consequence, of the 207 $\mathrm{O}(q^4,\phi^{2n})$ terms (obtained in the Section 5), there are (207 - 3)+113 = 317 linearly independent $\mathrm{O}(q^4)$ LECs. For on-shell nucleons, the LECs of 44 terms are fixed relative to lower order terms.

6 Application

We now discuss an application of the $\mathrm{O}(q^4)$ terms of Section 4 to evaluation of the contribution of the $\mathrm{O}(q^4)$ operator insertions to the "contact graph" of pion DCX (See Figs. 1 and 2) at threshold and assuming static nucleons. In [10] we included Fig 1, whose vertices, given in Fig 2, were taken to LO, which is $\mathrm{O}(q)$. In [6], the vertices were corrected (but only on-shell) to $\mathrm{O}(q^3)$. We will show that one gets no contribution from the $\mathrm{O}(q^4)$ terms (at threshold, and in the static limit of the nucleons). We will also discuss generalization of this result. The notations of [10] will be used in this section.

Let $p^\mu_{1,2}$ and $p^\mu_{3,4}$ be the 4-momenta of the incoming and outgoing nucleons (respectively), and $q^\mu_{1,2}$ the 4-momenta of the incoming and outgoing pions (respectively). The velocity parameters of the two participating nucleons are both chosen to have the static limit values, with only a non-vanishing time component , i.e. $v^\mu_1 = v^\mu_2 = (1,\vec{0})$. Also the nucleons will be treated as if they were on-shell (sometimes referred to as impulse approximation). In the HBChPT formalism, p^0 denotes only the contribution to the time component of the total nucleon momenta ($\equiv mv+p$), *in addition* to the rest mass energy m (for the choice of the nucleon velocity to possess only a non-zero time component). In the present case $p^0 = E_B$, which as stated above, we drop. Then if we go to the c.m. frame of the nucleons:

$$p^\mu_1 = (0,\vec{p});\ p^\mu_2 = (0,-\vec{p});\ p^\mu_3 = (0,\vec{p}');\ p^\mu_4 = (0,-\vec{p}');\ q^\mu_1 = q^\mu_2 = (M_\pi,\vec{0}). \tag{59}$$

(Note: The external pions are at zero kinetic energy [threshold].) Equation (59) implies:

$$v \cdot \mathcal{P} = \mathrm{S} \cdot q_{1,2} = \mathcal{P} \cdot q_{1,2} = 0, \tag{60}$$

where $\mathcal{P} \equiv p_1$ or p_3 or $p_1 - p_3$ In addition, one can show that:

$$[\Gamma_\mu^{(2)}, u_\nu^{(1)}]_+|_{\pi^0=0} = 0, \tag{61}$$

where the superscripts refer to the powers of ϕ.

One can show that all 113 terms of Table 2 (terms in table 1 will not contribute to the required vertices) will consist of one or more of the following terms (written symbolically):

$$\begin{aligned} &\mathrm{S} \cdot \partial\pi^+;\ v \cdot \partial\pi^-;\ v \cdot \partial\Big((\pi^+)^2\pi^-\Big); \\ &[\Gamma_\mu^{(2)}, u_\nu^{(1)}]_+|_{\pi^0}. \end{aligned} \tag{62}$$

In addition, using (59), in momentum space, some terms in Table 2 also consist of:

$$v \cdot q_{1,2}\mathrm{M}_\pi^2 + q_1 \leftrightarrow -q_2. \tag{63}$$

Using (60) and (61) in (62) and (63), one sees that one receives no contribution from any of the 113 terms of Table 2.

Thus, $\mathrm{O}(q^4)$ terms give no contribution to the contact DCX graph at threshold and in the impulse approximation (for the nucleons). One can in fact generalize this result. For this, one notes that because of parity constraints, L.C.-independent terms contributing to an $\mathrm{O}(\phi^{2n+1})$-vertex have to consist of an S_μ, and similarly, L.C-dependent terms contributing to an $\mathrm{O}(\phi^{2n+1})$-vertex have to be S_μ-independent. Further, no L.C. -dependent term will contribute to the contact graph. One way to see this is to note that for the $\bar{p}(\pi^+)^2\pi^- n$-vertex, in momentum space, L.C-dependent terms will consist of one of the following terms:

$$\begin{aligned} &\epsilon_{\mu\nu\rho\lambda}v^\mu p_1^\nu q_1^\rho q_2^\lambda; \\ &\epsilon_{\mu\nu\rho\lambda}v^\mu p_3^\nu q_1^\rho q_2^\lambda; \\ &\epsilon_{\mu\nu\rho\lambda}v^\mu p_1^\nu p_3^\rho q_1^\lambda; \\ &\epsilon_{\mu\nu\rho\lambda}v^\mu p_1^\nu p_3^\rho q_2^\lambda; \\ &\epsilon_{\mu\nu\rho\lambda}p_1^\mu p_3^\nu q_1^\rho q_2^\lambda. \end{aligned} \tag{64}$$

Using (59), one sees that all five terms in (64), vanish. One can similarly arrive at a similar conclusion for the $\bar{p}\pi^+ n$-vertex. Now, using (60), one sees

that the contribution of $\mathrm{O}(q^N)$ term to the $\bar{p}(\pi^+)^2\pi^- n$-vertex of the contact DCX graph can be written as:

$$(\mathrm{S}\cdot\mathcal{P})(v\cdot q_1)^{l_1}(v\cdot q_2)^{l_2}(q_1\cdot q_2)^{l_3}\mathrm{M}_\pi^{2l_4} + q_1 \leftrightarrow -q_2, \tag{65}$$

where $N = 1 + l_1 + l_2 + 2l_3 + 2l_4$. This implies that only those terms that correspond to $l_1 + l_2 \equiv$ even, i.e., terms in the Lagrangian of odd chiral orders, will contribute to the contact DCX graph vertices.

7 On-shell reduction

In this section, we discuss the derivation of the on-shell $\mathrm{O}(q^4)$ $\mathcal{L}_{\mathrm{HBChPT}}$, directly within HBChPT using the techniques of [1].

The main result obtained in [1] in the context of complete on-shell reduction within HBChPT was the following rule:

$$\begin{aligned}
&A-\text{type terms of the form }\bar{\mathrm{H}}\mathrm{S}\cdot\mathrm{D}\mathcal{O}\mathrm{H}\ +\ \text{h.c.}\\
&\text{or }\bar{\mathrm{H}}\mathrm{v}\cdot\mathrm{D}\mathcal{O}\mathrm{H}\ +\ \text{h.c.}\\
&\text{or }\bar{\mathrm{H}}\mathcal{O}^\mu\mathrm{D}_\mu\mathrm{H}\ +\ \text{h.c. can be eliminated}\\
&\text{except for }\mathcal{O}_\mu \equiv \left(i^{m_1+l_5+l_7+1},\ \text{or }\epsilon^{\nu\lambda\kappa\rho}\times\Omega\right)\times u_\mu\Lambda\\
&\text{with }l_1\geq 1, \Omega\equiv 1(i)\text{ for }(-)^{m_1+l_5+l_7+1} = -1(1),\\
&\text{or}\\
&\mathcal{O}_\mu \equiv \left(i^{m_1+l_5+l_7},\ \text{or }\epsilon^{\nu\lambda\kappa\rho}\times\Omega'\right)\times \mathrm{D}_\mu\Lambda,\\
&\text{with }l_1\geq 1, \Omega'\equiv 1(i)\text{ for }(-)^{m_1+l_5+l_7} = -1(1).
\end{aligned}\tag{66}$$

In (66)

$$\Lambda \equiv \prod_{i=1}^{M_1}\mathcal{V}_{\nu_i}\prod_{j=1}^{M_2}u_{\rho_j}(v\cdot u)^{l_1}u^{2l_2}\chi_+^{l_3}\chi_-^{l_4}([v\cdot\mathrm{D},\ ])^{l_5}(\mathrm{D}_\beta\mathrm{D}^\beta)^{l_6}(u_\alpha\mathrm{D}^\alpha)^{l_7}, \tag{67}$$

where $\mathcal{V}_{\nu_i}\equiv v_{\nu_i}$ or $\mathrm{D}_{\nu_\mathrm{i}}$. where $\mathcal{V}_{\nu_i}\equiv v_{\nu_i}$ or $\mathrm{D}_{\nu_\mathrm{i}}$. The number of D_{ν_i}s in (67) equals $m_1(\leq M_1)$. Assuming that Lorentz invariance, isospin symmetry, parity and hermiticity have been implemented, the choice of the factors of i in (66) automatically incorporates the phase rule (6). In (66), it is only the contractions of the building blocks that has been indicated. It is understood that all (anti-)commutators in the HBChPT Lagrangian are to be expanded

out until one hits the first D_μ, so that the A-type HBChPT term can be put in the form $\bar{\mathrm{H}}\mathcal{O}^\mu \mathrm{D}_\mu \mathrm{H}$ +h.c.

Using (66), we perform on-shell reduction of terms in Tables 1 and 2 . Terms that get eliminated are marked by an "E" and those that are not are marked by an "ON" in Tables 1 and 2 . Terms in Table 2 with $i = 211, 229, 244, 253, 261, 268, 269, 276, 277, 278, 305$ are marked by "ON′." This is because it is not the whole term, but only the on-shell "component" of these terms that do not get eliminated for on-shell nucleons. These terms can generically be written as $i[[u_\mu, u_\nu]_+, [u_\rho, \mathrm{D}_\lambda]_+]_+$,in which it is understood that the Lorentz indices are contracted either within themselves or by v_α and/or S_β. The on-shell "component" of these terms can be shown to be equal to $i[u_\rho, [\mathrm{D}_\lambda, [u_\mu, u_\nu]_+]]$.

The complete on-shell $\mathrm{O}(q^4)$ HBChPT Lagrangian can be shown to be given by:

$$\begin{aligned}
&\mathcal{L}^{(4)}_{\mathrm{HBChPT}} = A^{(4)} + \frac{1}{2\mathrm{m}}\gamma^0 B^{(2)\ \dagger}\gamma^0 B^{(2)} \\
&+\frac{1}{2\mathrm{m}}\left[\gamma^0 B^{(3)\ \dagger}\gamma^0 B^{(1)} + \gamma^0 B^{(1)\ \dagger}\gamma^0 B^{(3)}\right] \\
&-\frac{1}{4\mathrm{m}^2}\left[\gamma^0 B^{(2)\ \dagger}\gamma^0 C^{(1)} B^{(1)} + \gamma^0 B^{(1)\ \dagger}\gamma^0 C^{(1)} B^{(2)}\right] \\
&-\frac{1}{4\mathrm{m}^2}\gamma^0 B^{(1)\ \dagger}\gamma^0 C^{(2)} B^{(1)} + \frac{1}{8\mathrm{m}^3}\gamma^0 B^{(1)\ \dagger}\gamma^0 (C^{(1)})^2 B^{(1)}.
\end{aligned} \tag{68}$$

One can use the results of [1] for writing down on-shell B directly within HBChPT, but for the case at hand, we find it equally convenient to use the relativistic counterparts of $A^{(2)}$ and $A^{(3)}$ given in [1] for constructing on-shell $B^{(2)}$ and $B^{(3)}$. Of course the α_3 term (of (69)) has to be put in addition as γ^5 does not contribute to A.

Using

$$B^{(2)}_{\mathrm{OS}} = \alpha_1\gamma^5[v\cdot u, \mathrm{S}\cdot u] + \alpha_2\gamma^5[v\cdot u, \mathrm{S}\cdot u]_+ + \alpha_3\gamma^5\chi_- + i\alpha_4[v\cdot \mathrm{D}, v\cdot u], \tag{69}$$

(OS≡on-shell) one gets:

$$\begin{aligned}
&\frac{1}{2\mathrm{m}}\gamma^0 B^{(2)\ \dagger}\gamma^0 B^{(2)} = \\
&\alpha_1^2[v\cdot u, \mathrm{S}\cdot u]^2 - \alpha_2^2[v\cdot u, \mathrm{S}\cdot u]_+^2 + \alpha_3^2\chi_-^2 - \alpha_4^2[v\cdot \mathrm{D}, v\cdot u]^2 \\
&+\alpha_1\alpha_2\left(-\frac{1}{2}[[v\cdot u, u^\mu], [v\cdot u, u_\mu]_+] + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[v\cdot u, u_\mu], [v\cdot u, u_\nu]_+]_+\right) \\
&-\alpha_1\alpha_3[[v\cdot u, \mathrm{S}\cdot u], \chi_-] + i\alpha_1\alpha_4[[v\cdot u, \mathrm{S}\cdot u], [v\cdot \mathrm{D}, v\cdot u]]_+ \\
&-i\alpha_2\alpha_4[[v\cdot u, \mathrm{S}\cdot u]_+, [v\cdot \mathrm{D}, v\cdot u]] + i\alpha_3\alpha_4[\chi_-, [v\cdot \mathrm{D}, v\cdot u]]_+.
\end{aligned} \tag{70}$$

The set $\{\alpha_i\}$ can be related to the set $\{a_i\}$ of [5].

Using (69) and $C^{(1)}_{\mathrm{OS}}$, and eliminating all terms proportional to the non-relativistic eom by field redefinition of H, one gets:

$$
\begin{aligned}
&-\frac{1}{4\mathrm{m}^2}\gamma^0 B^{(2)\ \dagger}\gamma^0 C^{(1)} B^{(1)} + \mathrm{h.c.} \\
&= -\frac{1}{4\mathrm{m}^2}\Bigg[2\alpha_1\Bigg(-\frac{1}{16}[v\cdot u, u^\mu]^2 + ig_A^0[v\cdot u, u_\mu]\mathrm{D}^\mu \mathrm{S}\cdot u + \frac{i}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu][v\cdot u, u_\nu] \\
&+\frac{g_A^0}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot u, u_\mu]\mathrm{D}_\nu u_\lambda - \frac{g_A^0}{2}\Bigg[\frac{i}{2}[v\cdot u, \mathrm{S}\cdot u]\mathrm{D}_\mu u^\mu - \frac{i}{2}[v\cdot u, \mathrm{S}\cdot u][v\cdot \mathrm{D}, v\cdot u] \\
&+\frac{g_A^0}{8}[v\cdot u, u_\mu]v\cdot u u_\mu + \frac{ig_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]v\cdot u u_\nu\Bigg] \\
&-\frac{ig_A^0}{2}[v\cdot u, \mathrm{S}\cdot u][v\cdot \mathrm{D}, v\cdot u] + \frac{g_A^{0\,2}}{2}\Bigg[-\frac{1}{4}[v\cdot u, u_\mu]v\cdot u u^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]v\cdot u u_\nu\Bigg] \\
&-2g_A^0\Bigg[-\frac{i}{4}[v\cdot u, u_\mu]u^\mu \mathrm{S}\cdot \mathrm{D} + \frac{1}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot u, u_\mu]u_\nu \mathrm{D}_\lambda \\
&-\frac{i}{4}[v\cdot u, \mathrm{S}\cdot u]u_\mu \mathrm{D}^\mu + \frac{g_A^0}{16}[v\cdot u, u_\mu]v\cdot u u^\mu \\
&-\frac{g_A^0}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]v\cdot u u_\nu + \frac{1}{4}[v\cdot u, u_\mu]\mathrm{S}\cdot u \mathrm{D}^\mu\Bigg] \\
&+\frac{g_A^{0\,2}}{8}[v\cdot u, u_\mu]u^\mu v\cdot u - \frac{ig_A^{0\,2}}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]u_\nu v\cdot u\Bigg) \\
&-\alpha_2\Bigg(-\frac{1}{16}[v\cdot u, u^\mu]_+[v\cdot u, u_\mu] + ig_A^0[v\cdot u, u_\mu]_+\mathrm{D}^\mu \mathrm{S}\cdot u + \frac{i}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]_+[v\cdot u, u_\nu] \\
&+\frac{g_A^0}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot u, u_\mu]_+\mathrm{D}_\nu u_\lambda - \frac{g_A^0}{2}\Bigg[\frac{i}{2}[v\cdot u, \mathrm{S}\cdot u]_+\mathrm{D}_\mu u^\mu - \frac{i}{2}[v\cdot u, \mathrm{S}\cdot u]_+[v\cdot \mathrm{D}, v\cdot u] \\
&+\frac{g_A^0}{8}[v\cdot u, u_\mu]_+v\cdot u u_\mu + \frac{ig_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]_+v\cdot u u_\nu\Bigg] \\
&-\frac{ig_A^0}{2}[v\cdot u, \mathrm{S}\cdot u]_+[v\cdot \mathrm{D}, v\cdot u] + \frac{g_A^{0\,2}}{2}\Bigg[-\frac{1}{4}[v\cdot u, u_\mu]_+v\cdot u u^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]_+v\cdot u u_\nu\Bigg] \\
&-2g_A^0\Bigg[-\frac{i}{4}[v\cdot u, u_\mu]_+u^\mu \mathrm{S}\cdot \mathrm{D} + \frac{1}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot u, u_\mu]_+u_\nu \mathrm{D}_\lambda \\
&-\frac{i}{4}[v\cdot u, \mathrm{S}\cdot u]_+u_\mu \mathrm{D}^\mu + \frac{g_A^0}{16}[v\cdot u, u_\mu]_+v\cdot u u^\mu \\
&-\frac{g_A^0}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u, u_\mu]_+v\cdot u u_\nu + \frac{1}{4}[v\cdot u, u_\mu]_+\mathrm{S}\cdot u \mathrm{D}^\mu\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{g_A^{0\,2}}{8}[v\cdot u,u_\mu]_+u^\mu v\cdot u-\frac{ig_A^{0\,2}}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot u,u_\mu]_+u_\nu v\cdot u\Big)\\
&+\alpha_3\Big(2\chi_-v\cdot \mathrm{D}\mathrm{S}\cdot \mathrm{D}-i\frac{g_A^0}{2}\chi_-[v\cdot \mathrm{D},v\cdot u]+\frac{g_A^{0\,2}}{2}\chi_-v\cdot u\mathrm{S}\cdot u\\
&+g_A^0\chi_-\Big[\frac{1}{4}v\cdot u\mathrm{S}\cdot u+\frac{i}{4}u_\mu \mathrm{D}^\mu+\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu \mathrm{D}_\nu\Big]\Big)\\
&+\alpha_4\Big(\frac{i}{2}[v\cdot \mathrm{D},v\cdot u][v\cdot u,\mathrm{S}\cdot u]-2g_A^0\Big[\frac{1}{4}[v\cdot \mathrm{D},v\cdot u]^2-\frac{1}{4}[v\cdot \mathrm{D},v\cdot u]\mathrm{D}_\mu u^\mu\\
&+\frac{ig_A^0}{4}[v\cdot \mathrm{D},v\cdot u]v\cdot u\mathrm{S}\cdot u+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot \mathrm{D},v\cdot u]\mathrm{D}_\mu u_\nu\Big]\\
&+\frac{g_A^0}{2}[v\cdot \mathrm{D},v\cdot u]^2+\frac{ig_A^{0\,2}}{2}[v\cdot \mathrm{D},v\cdot u]v\cdot u\mathrm{S}\cdot u\\
&+2g_A^0\Big[\frac{ig_a^0}{4}[v\cdot \mathrm{D},v\cdot u]v\cdot u\mathrm{S}\cdot u-\frac{1}{4}[v\cdot \mathrm{D},v\cdot u]u_\mu \mathrm{D}^\mu\\
&+\frac{i}{2}v_\rho \mathrm{S}_\lambda[v\cdot \mathrm{D},v\cdot u]u_\mu \mathrm{D}_\nu\Big]-\frac{ig_A^{0\,2}}{2}[v\cdot \mathrm{D},v\cdot u]\mathrm{S}\cdot uv\cdot u\Big)\\
&+\mathrm{h.c.}\Big]
\end{aligned}
\tag{71}
$$

Similarly, using:

$$
C_{\mathrm{OS}}^{(2)}=-i\alpha_1\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda-\alpha_2(v\cdot u)^2-\alpha_5u^2-\alpha_6\chi_+,
\tag{72}
$$

and

$$
B^{(1)}=-2i\gamma^5\mathrm{S}\cdot \mathrm{D}-\frac{g_A^0}{2}\gamma^5v\cdot u,
\tag{73}
$$

and eliminating all terms proportional to the nonrelativistic eom by field redefinition of H, one sees that:

$$
\begin{aligned}
&-\frac{1}{4\mathrm{m}^2}\gamma^0B^{(1)\;\dagger}\gamma^0C^{(2)}B^{(1)}\\
&=-\frac{1}{4\mathrm{m}^2}\Big[\alpha_1\Big(i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{D}_\lambda u_\mu u_\nu \mathrm{S}\cdot \mathrm{D}-i\frac{g_A^0}{4}\mathrm{S}\cdot u[u^\mu,v\cdot u]\mathrm{D}_\mu-\frac{g_A^0}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho u_\mu[u_\lambda,v\cdot u]\mathrm{D}_\nu\\
&+\frac{ig_A^0}{4}(-u^\mu[\mathrm{S}\cdot u,v\cdot u]\mathrm{D}_\mu+u^\mu[u_\mu,v\cdot u]\mathrm{S}\cdot \mathrm{D})\\
&-\frac{g_A^{0\,2}}{4}v\cdot u(-\frac{1}{4}[u_\mu,v\cdot u]u^\mu+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\mu,v\cdot u]u_\nu)
\end{aligned}
$$

$$+\Big[\frac{g_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{D}_\lambda u_\mu u_\nu v\cdot u - \frac{g_A^0}{2}\Big(-i\mathrm{D}^\mu \mathrm{S}\cdot u u_\mu v\cdot u$$
$$-g_A^0[-\frac{1}{4}u^2(v\cdot u)^2 + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu u_\nu (v\cdot u)^2]$$
$$+g_A^0(-\frac{1}{4}[u_\mu, v\cdot u]u^\mu v\cdot u + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu v\cdot u u_\nu v\cdot u) + i\mathrm{D}^\mu u_\mu \mathrm{S}\cdot u v\cdot u\Big) + \text{h.c.}\Big]$$
$$+\frac{g_A^{0\,2}}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda v\cdot u u_\mu u_\nu v\cdot u\Big)$$
$$+\alpha_2\Big(g_A^{0\,2}\Big[\frac{1}{4}(v\cdot u)^2 - \frac{1}{4}u_\mu (v\cdot u)^2 u_\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu (v\cdot u)^2 u_\nu\Big]$$
$$+ig_A^0[\mathrm{S}\cdot \mathrm{D}, (v\cdot u)^3]_+ + \frac{g_A^{0\,2}}{4}(v\cdot u)^4 + \mathrm{D}_\mu (v\cdot u)^2 \mathrm{D}^\mu - 2i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\mu (v\cdot u)^2 \mathrm{D}_\nu\Big)$$
$$+\alpha_5\Big(g_A^{0\,2}\Big[\frac{1}{4}v\cdot u u^2 v\cdot u - \frac{1}{4}u_\mu u^2 u^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu u^2 u_\nu\Big]$$
$$+\mathrm{D}_\mu u^2 \mathrm{D}^\mu - 2i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\mu u^2 \mathrm{D}_\nu + (ig_A^0 \mathrm{S}\cdot \mathrm{D} u^2 v\cdot u + \text{h.c.}) + \frac{g_A^{0\,2}}{4}v\cdot u u^2 v\cdot u\Big)$$
$$+\alpha_6\Big(g_A^{0\,2}\Big[\frac{1}{4}v\cdot u\chi_+ v\cdot u - \frac{1}{4}u_\mu \chi_+ u^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu \chi_+ u_\nu\Big]$$
$$+\mathrm{D}_\mu \chi_+ \mathrm{D}^\mu - 2i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\mu \chi_+ \mathrm{D}_\nu$$
$$+(ig_A^0 v\cdot u\chi_+ \mathrm{S}\cdot \mathrm{D} + \text{h.c.}) + \frac{g_A^{0\,2}}{4}v\cdot u\chi_+ v\cdot u\Big)\Big] \tag{74}$$

Further, the $\mathrm{O}(q^4)$ terms, independent of any undetermined LECs are given by (after elimination of the terms proportional to the nonrelativistic eom, by field-redefinition of H)

$$\frac{1}{8\mathrm{m}^3}\gamma^0 B^{(1)\ \dagger}\gamma^0 (C^{(1)})^2 B^{(1)}$$
$$= \frac{1}{8\mathrm{m}^3}\Big[3g_A^{0\,2}\Big[\frac{1}{16}(v\cdot u)^4 + \frac{1}{16}u^4$$
$$+\frac{1}{16}\Big(-u^\mu u^\nu u_\mu u_\nu + v\cdot u u^\mu v\cdot u u_\mu + u_\mu v\cdot u u^\mu v\cdot u - v\cdot u u^2 v\cdot u - u^\mu (v\cdot u)^2 u_\mu + u^\mu u^2 u_\mu\Big)$$
$$+\frac{i}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho u_\mu u_\nu [u_\lambda, \mathrm{S}\cdot u] - \frac{1}{16}[(v\cdot u)^2, u^2]_+ - \frac{i}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda [(u^2 - (v\cdot u)^2), u_\mu u_\nu]_+\Big]$$
$$-g_A^{0\,2}\Big[ig_A^0\Big(\frac{1}{4}(v\cdot u)^2 - \frac{1}{4}u^2\Big)[v\cdot \mathrm{D}, \mathrm{S}\cdot u] + \frac{g_A^0}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho u_\mu u_\nu [v\cdot \mathrm{D}, u_\lambda]$$

$$
\begin{aligned}
&+i\frac{g_A^0}{4}\Big(-[\mathrm{S}\cdot u,u_\mu][v\cdot\mathrm{D},u^\mu]+[\mathrm{S}\cdot u,v\cdot u][v\cdot\mathrm{D},v\cdot u]\Big)\\
&+\frac{ig_A^0}{2}\mathrm{S}\cdot u\Big([v\cdot\mathrm{D},v\cdot u]v\cdot u-[v\cdot\mathrm{D},u^\mu]u_\mu\Big)\\
&+\frac{g_A^0}{4}\Big(-u^\mu[v\cdot\mathrm{D},\mathrm{S}\cdot u]v\cdot u+v\cdot u[v\cdot\mathrm{D},\mathrm{S}\cdot u]v\cdot u\\
&-v\cdot u[v\cdot\mathrm{D},v\cdot u]\mathrm{S}\cdot u+u^\mu[v\cdot\mathrm{D},u_\mu]\mathrm{S}\cdot u\Big)\\
&+\frac{1}{4}v\cdot u[v\cdot\mathrm{D},[v\cdot\mathrm{D},v\cdot u]]-\frac{u^\mu}{4}[v\cdot\mathrm{D},[v\cdot\mathrm{D},u_\mu]]+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu[v\cdot\mathrm{D},[v\cdot\mathrm{D},u_\nu]]\Big]\\
&-\frac{1}{16}[u_\mu,v\cdot u]^2+\frac{ig_A^0}{4}[\mathrm{S}\cdot u,\mathrm{D}_\mu[v\cdot u,u^\mu]]_+\\
&+2i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(\frac{\mathrm{S}_\lambda}{16}[u_\mu,v\cdot u][v\cdot u,u_\nu]-\frac{ig_A^0}{16}[u_\lambda,\mathrm{D}_\mu[v\cdot u,u_\nu]]_+\Big)\\
&-\frac{ig_A}{4}\Big(-[u^\mu,\mathrm{S}\cdot\mathrm{D}[v\cdot u,u_\mu]]_+-[v\cdot u,v\cdot u\mathrm{D}[v\cdot u,\mathrm{S}\cdot u]]_++[u^\mu,\mathrm{D}_\mu,[v\cdot u,\mathrm{S}\cdot u]]_+\Big)\\
&+ig_A^0\Big(\frac{1}{4}[\mathrm{S}\cdot u,v\cdot u][v\cdot\mathrm{D},v\cdot u]-\frac{ig_A^0}{4}[v\cdot u,v\cdot\mathrm{D}]^2-\frac{{g_A^0}^2}{4}\mathrm{S}\cdot uv\cdot u[v\cdot\mathrm{D},v\cdot u]\\
&-\frac{ig_A^0}{4}u_\mu\mathrm{D}^\mu[v\cdot\mathrm{D},v\cdot u]-\frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu\mathrm{D}_\nu[v\cdot\mathrm{D},v\cdot u]-\frac{{g_A^0}^2}{4}\mathrm{S}\cdot u[v\cdot\mathrm{D},v\cdot u]v\cdot u\\
&-\frac{ig_A^0}{4}\mathrm{D}_\mu[v\cdot\mathrm{D},v\cdot u]u^\mu-\frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\mu[v\cdot\mathrm{D},v\cdot u]u_\nu+\mathrm{h.c.}\Big)\\
&-\frac{{g_A^0}^2}{4}\Big(-[v\cdot u,v\cdot\mathrm{D}]^2+ig_A^0[\mathrm{S}\cdot u,v\cdot u[v\cdot\mathrm{D},v\cdot u]]_+\Big)\\
&-ig_A^0\Big(\frac{1}{4}[u_\mu,v\cdot u]\mathrm{S}\cdot u\mathrm{D}^\mu+\frac{ig_A^0}{4}v\cdot u\mathrm{D}_\mu v\cdot u\mathrm{D}^\mu-\frac{ig_A^0}{4}u_\mu\mathrm{D}_\nu u^\mu\mathrm{D}^\nu\\
&-\frac{g_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\kappa u_\mu\mathrm{D}^\kappa u_\nu+\mathrm{h.c.}\Big)\\
&-{g_A^0}^3\Big(\frac{1}{4}[\mathrm{S}\cdot u,[v\cdot\mathrm{D},\{(v\cdot u)^2-u^2\}]]-\frac{i}{16}\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\lambda,[v\cdot\mathrm{D},[u_\mu,u_\nu]]]\\
&+\frac{1}{4}[u^\mu,[v\cdot\mathrm{D},[\mathrm{S}\cdot u,u_\mu]]]_+-\frac{1}{4}[v\cdot u,[v\cdot\mathrm{D},[\mathrm{S}\cdot u,v\cdot u]]]_+\Big)\\
&+\frac{g_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(ig_A^0\mathrm{S}\cdot u\mathrm{D}_\mu u_\lambda\mathrm{D}_\nu+\frac{1}{4}[u_\mu,v\cdot u]u_\lambda\mathrm{D}_\nu+\mathrm{h.c.}\Big)\\
&+{g_A^0}^2\Big[\frac{{g_A^0}^2}{4}(v\cdot u)^2[\frac{1}{4}(v\cdot u)^2-\frac{1}{4}u^2+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu u_\nu]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{ig_A^0}{4}\Big([\mathrm{S}\cdot u, v\cdot \mathrm{D}](v\cdot u)^2+\frac{ig_A^0}{4}(v\cdot u)^4-\frac{ig_A^0}{4}u_\mu(v\cdot u)^2uu^\mu \\
&-\frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu(v\cdot u)^2u_\nu+\mathrm{S}\cdot uv\cdot u[v\cdot\mathrm{D},v\cdot u]\Big) \\
&-\frac{g_A^0}{2}\mathrm{S}\cdot u\mathrm{D}_\mu u^\mu v\cdot u-\frac{1}{16}[u^\mu,v\cdot u]u^\mu v\cdot u-\frac{ig_A^0}{2}\mathrm{D}_\mu u^\mu[v\cdot\mathrm{D},v\cdot u] \\
&-\frac{ig_A^0}{2}\mathrm{D}_\mu u^\mu v\cdot u\mathrm{S}\cdot u+\mathrm{h.c.}\Big] \\
&-\frac{{g_A^0}^3}{4}\Big(iv\cdot u\mathrm{S}\cdot u[v\cdot\mathrm{D},v\cdot u]+\mathrm{h.c.}-\frac{g_A^0}{2}(v\cdot u)^4+\frac{g_A^0}{2}v\cdot uu^\mu v\cdot uu_\mu \\
&-ig_A^0\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda v\cdot uu_\mu v\cdot uu_\nu\Big) \\
&+\frac{{g_A^0}^2}{4}\Big[-{g_A^0}^2\Big(\frac{1}{4}v\cdot u[(v\cdot u)^2-u^2]v\cdot u-\frac{1}{4}u_\mu[(v\cdot u)^2-u^2]u^\mu \\
&+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu[(v\cdot u)^2-u^2]u_\nu\Big)-\mathrm{D}_\mu[(v\cdot u)^2-u^2]\mathrm{D}^\mu+2i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}_\mu[(v\cdot u)^2-u^2]\mathrm{D}_\nu\Big] \\
&-(\frac{i{g_A^0}^3}{4}\mathrm{S}\cdot\mathrm{D}[(v\cdot u)^2-u^2]v\cdot u+\mathrm{h.c.})-\frac{{g_A^0}^4}{8}v\cdot u[(v\cdot u)^2-u^2]v\cdot u-\frac{ig_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{D}_\lambda u_\mu u_\nu v\cdot u \\
&+\frac{ig_A^0}{2}\Big[(-\frac{i}{2}\mathrm{D}^\mu u^\nu u_\mu\mathrm{D}_\nu+\mathrm{h.c.})-\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\mathrm{D}^\kappa[u_\mu,u_\kappa]\mathrm{D}_\nu+(\frac{1}{2}g_A^0\mathrm{D}^\mu[v\cdot u,u_\mu]_+\mathrm{S}\cdot u+\mathrm{h.c.}) \\
&-\frac{ig_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho u_\lambda[v\cdot u,u_\mu]\mathrm{D}_\nu+\frac{g_A^0}{2}\Big(-u^\mu[v\cdot u,\mathrm{S}\cdot u]\mathrm{D}_\mu-\frac{ig_A^0}{4}v\cdot u[v\cdot u,u^\mu]u_\mu \\
&-\frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda v\cdot u[v\cdot u,u_\mu]u_\nu+u^\mu[v\cdot u,u_\mu]\mathrm{S}\cdot\mathrm{D}\Big)\Big] \\
&+\frac{g_A^0}{2}\Big[-\frac{1}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{D}_\lambda u_\mu u_\nu v\cdot u+\frac{i}{2}\Big(-\mathrm{D}_\mu[\mathrm{S}\cdot u,u_\mu]v\cdot u \\
&+ig_A^0[-\frac{1}{4}u^2(v\cdot u)^2+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu u_\nu(v\cdot u)^2] \\
&-g_A^0[-\frac{1}{4}u_\mu v\cdot uu^\mu v\cdot u+\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda u_\mu v\cdot uu_\nu v\cdot u]\Big)+\mathrm{h.c.}\Big] \\
&-\frac{i{g_A^0}^3}{8}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda v\cdot uu_\mu u_\nu v\cdot u.
\end{aligned}
\tag{75}
$$

Using:

$$
\begin{aligned}
&B^{(3)}_{\mathrm{OS}}=\beta_1\gamma^5[u^2,v\cdot u]_++\beta_2\gamma^5u^\mu v\cdot uu_\mu \\
&+i\beta_3\epsilon^{\mu\nu\rho\lambda}\gamma^5\mathrm{S}_\mu[[u_\nu,u_\rho],u_\lambda]_++\beta_4\gamma^5[v\cdot u,\chi_+]_+
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_5\gamma^5(v\cdot u)^3+i\beta_6\epsilon^{\mu\nu\rho\lambda}\gamma^5 v_\rho \mathrm{S}_\lambda[v\cdot u,[u_\mu,u_\nu]]_+ \\
&+i\beta_7\epsilon^{\mu\nu\rho\lambda}\gamma^5 v_\rho \mathrm{S}_\lambda[u_\mu,[v\cdot u,u_\nu]]_+ + i\beta_8\gamma^5[v\cdot u,[\mathrm{S}\cdot \mathrm{D},v\cdot u]] \\
&+i\beta_9\gamma^5[u^\mu,[\mathrm{S}\cdot\mathrm{D},u_\mu]]+i\beta_{10}\gamma^5\Big([v\cdot u,[v\cdot\mathrm{D},\mathrm{S}\cdot u]]_+ - [\mathrm{S}\cdot u,[v\cdot \mathrm{D},v\cdot u]]_+\Big) \\
&+i\beta_{11}\gamma^5[u^\mu,[\mathrm{S}\cdot u,\mathrm{D}_\mu]_+]_+ + i\beta_{12}\gamma^5[u^\mu,[\mathrm{S}\cdot u,\mathrm{D}_\mu]] \\
&+i\beta_{13}\gamma^5\Big([v\cdot u,[\mathrm{S}\cdot\mathrm{D},v\cdot u]]_+ - [\mathrm{S}\cdot u,[v\cdot\mathrm{D},v\cdot u]]_+\Big) \\
&+\beta_{14}\gamma^5[\chi_-,\mathrm{S}\cdot u]+i\beta_{15}\gamma^5[v\cdot u,[v\cdot\mathrm{D},\mathrm{S}\cdot u]] \\
&+i\beta_{16}\gamma^5[\mathrm{S}\cdot u,[v\cdot\mathrm{D},v\cdot u]], \qquad (76)
\end{aligned}
$$

$\frac{1}{2\mathrm{m}}\Big[\gamma^0 B^{(3)\ \dagger}\gamma^0 B^{(1)}+\gamma^0 B^{(1)\ \dagger}\gamma^0 B^{(3)}\Big]$ and eliminating all terms proportional to the nonrelativistic eom by field redefinition of H, one gets:

$$
\begin{aligned}
&\frac{1}{2\mathrm{m}}\Bigg(\beta_1\Big[-2i[[u^2,v\cdot u]_+,\mathrm{S}\cdot\mathrm{D}]_+ - \frac{g_A^0}{2}[[u^2,v\cdot u]_+,v\cdot u]_+\Big] \\
&+\beta_2\Big[-2i[u^\mu v\cdot u u_\mu,\mathrm{S}\cdot\mathrm{D}]_+ + g_A^0[u^\mu v\cdot u u_\mu, v\cdot u]_+\Big] \\
&+\beta_3\Big[i\frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}[[[u_\nu,u_\rho],u_\lambda]_+,\mathrm{S}\cdot u]_+ \\
&+2i[\mathrm{D}_\mu,\Big([\mathrm{S}\cdot u,[u_\mu,v\cdot u]]_+ - [v\cdot u,[u^\mu,\mathrm{S}\cdot u]]_+] + [u^\mu,[[v\cdot u,\mathrm{S}\cdot u]]_+\Big)]\Big] \\
&+\beta_4\Big[-2i[\mathrm{S}\cdot\mathrm{D},[v\cdot u,\chi_+]_+]_+ \\
&-\frac{g_A^0}{2}[v\cdot u,[v\cdot u,\chi_+]_+]_+\Big] + \beta_5\Big[-2i[\mathrm{S}\cdot\mathrm{D},(v\cdot u)^3]_+ - \frac{g_A^0}{2}(v\cdot u)^4\Big] \\
&+\beta_6\Big[\frac{1}{2}[-\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\lambda,[v\cdot u,[u_\mu,u_\nu]]_+]_+ \\
&-i\frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[v\cdot u,[v\cdot u,[u_\mu,u_\nu]]_+]_+ - 2i[\mathrm{D}_\mu,[v\cdot u,[\mathrm{S}\cdot u,u^\mu]]_+]\Big] \\
&+\beta_7\Big[-\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\lambda,[u_\mu,[v\cdot u,u_\nu]]_+]_+ - i[\mathrm{D}_\mu,[u^\mu,[\mathrm{S}\cdot u,v\cdot u]]_+] \\
&+\frac{g_A^0}{2}\Big(-\frac{1}{2}[[u_\mu,v\cdot u],v\cdot u]_+,u^\mu] + i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[[[u_\mu,v\cdot u],v\cdot u]_+u_\nu]_+\Big) \\
&-\frac{ig_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[v\cdot u,[u_\mu,[v\cdot u,u_\nu]]_+]_+\Big]
\end{aligned}
$$

$$\beta_8\Big[-\frac{1}{2}[\mathrm{D}^\mu,[v\cdot u,[\mathrm{D}_\mu,v\cdot u]]]_+ + \frac{ig_A^0}{2}[\mathrm{S}\cdot u,[v\cdot u,[v\cdot \mathrm{D},v\cdot u]]]_+$$
$$-i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[v\cdot u,[\mathrm{D}_\mu,v\cdot u]]] - i\frac{g_A^0}{2}[v\cdot u,[v\cdot u,[\mathrm{S}\cdot \mathrm{D},v\cdot u]]]_+\Big]$$
$$\beta_9\Big[-\frac{1}{2}[\mathrm{D}^\mu,[u_\nu,[\mathrm{D}_\mu,u^\nu]]]_+ + \frac{ig_A^0}{2}[\mathrm{S}\cdot u,[u_\mu,[v\cdot \mathrm{D},u^\mu]]]_+ - i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\kappa,[\mathrm{D}_\mu,u^\kappa]]]$$
$$-i\frac{g_A^0}{2}[v\cdot u,[u_\mu,[\mathrm{S}\cdot \mathrm{D},u^\mu]]]_+ + \beta_{10}\Big[\frac{1}{2}[\mathrm{D}^\mu,[v\cdot u,[v\cdot \mathrm{D},u_\mu]]_+]$$
$$-\frac{ig_A^0}{2}[v\cdot u,[\mathrm{S}\cdot u,[v\cdot \mathrm{D},v\cdot u]]_+] - \frac{1}{2}[\mathrm{D}_\mu,[u^\mu,[v\cdot \mathrm{D},v\cdot u]]_+]$$
$$+i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\nu,[v\cdot u,[v\cdot \mathrm{D},u_\mu]]_+]_+ - [\mathrm{D}_\nu,[u_\mu,[v\cdot \mathrm{D},v\cdot u]]_+]_+\Big)$$
$$-\frac{1}{2}[\mathrm{D}_\mu,[u^\mu,[v\cdot \mathrm{D},v\cdot u]]_+] - \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\mu,[v\cdot \mathrm{D},v\cdot u]]_+]_+$$
$$+i\frac{g_A^0}{2}\Big([v\cdot u,[v\cdot u[v\cdot \mathrm{D},\mathrm{S}\cdot u]]_+] + [v\cdot u,[\mathrm{S}\cdot u,[v\cdot \mathrm{D},v\cdot u]]_+]\Big)\Big]$$
$$+\beta_{11}\Big[-\frac{1}{2}[\mathrm{D}_\nu,[u^\mu,[\mathrm{D}_\mu,u^\nu]_+]_+]_+ - i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u^\kappa,[u_\mu,\mathrm{D}_\kappa]_+]_+]$$
$$+\frac{ig_A^0}{2}[\mathrm{S}\cdot u,[u^\mu,[v\cdot u,\mathrm{D}_\mu]_+]_+]_+ + \frac{ig_A^0}{2}[\mathrm{S}\cdot u,[u^\mu,[v\cdot u,\mathrm{D}_\mu]]]_+$$
$$-i\frac{g_A^0}{2}[v\cdot u,[u^\mu,[\mathrm{S}\cdot u,\mathrm{D}_\mu]_+]_+]_+\Big]$$
$$+\beta_{12}\Big[-\frac{1}{2}[\mathrm{D}^\nu,[u^\mu,[u_\nu,\mathrm{D}_\mu]]]_+ - i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u^\kappa,[u_\mu,\mathrm{D}_\kappa]]]$$
$$-\frac{ig_A^0}{2}[v\cdot u,[u^\mu,[\mathrm{S}\cdot u,\mathrm{D}^\mu]]]_+\Big]$$
$$+\beta_{13}\Big[\frac{1}{2}[\mathrm{D}^\mu,[v\cdot u,[\mathrm{D}_\mu,v\cdot u]]_+] + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[v\cdot u,[\mathrm{D}_\mu,v\cdot u]]_+]_+$$
$$-\frac{1}{2}[\mathrm{D}_\mu,[u^\mu,[v\cdot \mathrm{D},v\cdot u]]_+] - \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\mu,[v\cdot \mathrm{D},v\cdot u]]_+]_+$$
$$+\frac{ig_A^0}{2}[v\cdot u,[v\cdot u,[\mathrm{S}\cdot \mathrm{D},v\cdot u]]_+] - \frac{ig_A^0}{2}[v\cdot u,[\mathrm{S}\cdot u,[v\cdot \mathrm{D},v\cdot u]]_+]\Big]$$
$$+\beta_{14}\Big[i[\mathrm{D}_\mu,[\chi_-,u^\mu]]_+ + g_A^0[\mathrm{S}\cdot u,[\chi_-,v\cdot u]]_+ - \epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[\chi_-,u_\mu]]$$
$$-\frac{g_A^0}{2}[v\cdot u,[\chi_-,\mathrm{S}\cdot u]]_+\Big] + \beta_{15}\Big[-\frac{1}{2}[\mathrm{D}_\mu,[v\cdot u,[v\cdot \mathrm{D},u^\mu]]]_+$$

$$
\begin{aligned}
&+\frac{ig_A^0}{2}[\mathrm{S}\cdot u,[v\cdot u,[v\cdot \mathrm{D},v\cdot u]]]_+ - i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[v\cdot u,[v\cdot \mathrm{D},u_\mu]]]\\
&-\frac{ig_A^0}{2}[v\cdot u,[v\cdot u,[v\cdot \mathrm{D},\mathrm{S}\cdot u]]]_+\Big] + \beta_{16}\Big[-\frac{1}{2}[\mathrm{D}_\mu,[u^\mu,[v\cdot \mathrm{D},v\cdot u]]]_+\\
&-g_A^0[\mathrm{S}\cdot u,[v\cdot u,[v\cdot \mathrm{D},v\cdot u]]]_+ - i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\nu,[u_\mu,[v\cdot \mathrm{D},v\cdot u]]]\\
&-\frac{ig_A^0}{2}[v\cdot u,[\mathrm{S}\cdot u,[v\cdot \mathrm{D},v\cdot u]]]_+\Big]\Big).
\end{aligned}
\tag{77}
$$

The set $\{\beta_i\}$ can be related to the set $\{b_i\}$ of [5].

So, the final result is:

$$
\mathcal{L}^{(4)}_{\mathrm{HBChPT}}(\mathrm{OS}) = A^{(4)}(\mathrm{ON\ or\ ON}') + (70) + (71) + (74) + (75) + (77).
\tag{78}
$$

8 Conclusion

A complete list of $\mathrm{O}(q^4)$ terms for off-shell nucleons was obtained *working within HBChPT* using a phase rule obtained in [1], along with reductions from algebraic identities and reparameterization invariance. We also obtain the on-shell $\mathrm{O}(q^4)$ terms, again within the framework of HBChPT. For this paper, we set the external fields to zero and assume isospin symmetry. For off-shell nucleons, one gets a total of 207 $\mathcal{O}(q^4, \phi^{2n})$ terms (given in Table 1) the LECs of three of which (given in Table 4) are fixed relative to some $\mathrm{O}(q^{1,2,3})$ terms, and 113 $\mathrm{O}(q^4, \phi^{2n+1})$ terms (given in Table 2). Of the total of 320 terms, 230 are finite. For on-shell nucleons, the LECs of 44 terms (given in (58)) are fixed relative to those of lower order terms. As an application of the off-shell $\mathrm{O}(q^4)$ list, we showed that none of the $\mathrm{O}(q^4)$ terms contribute to the contact graph of pion DCX at threshold and assuming static nucleons. In fact we argued that at threshold and for static nulceons, only terms of even chiral orders will contribute to the contact graph (of pion DCX). For future work, one could use (6) for the construction of $\mathrm{O}(q^4)$ terms including external fields and assuming isospin symmetry violation, within HBChPT. We also mention that the authors of [7] and M.Mojzis are working on constructing the full $\mathrm{O}(q^4)$ HBChPT Lagrangian, but starting from the BChPT Lagrangian.

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A

In this appendix, we show how to obtain a set of linearly independent terms from (16).

Using (9) and (12), one obtains (a) the following four identities in nine terms:

$$
\begin{aligned}
&[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]]-[[\mathrm{D}_\mu,\mathrm{D}_\nu],[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]_+=-[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]_+]_+\\
&-[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]]-[[u_\mu,u_\nu],[u^\mu,u^\nu]]_+=-\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u^\mu,u^\nu]]]_+\\
&[u_\mu,[u_\nu,[u^\mu,u^\nu]_+]]-[[u_\mu,u_\nu]_+,[u^\mu,u^\nu]_+]_+=-[u_\mu,[u_\nu,[u^\mu,u^\nu]_+]_+]_+\\
&-[u_\mu,[u_\nu,[u^\mu,u^\nu]_+]]-[[u_\mu,u_\nu],[u^\mu,u^\nu]]_+=-[u_\mu,[u_\nu,[u^\mu,u^\nu]]]_+, \qquad \text{(A1)}
\end{aligned}
$$

and

(b) the following two identities in five terms:

$$
\begin{aligned}
&[\mathrm{D}]_\mu,[\mathrm{D}_\nu,[u^\mu,u^\nu]_+]]-[[\mathrm{D}_\mu,\mathrm{D}_\nu]_+,[u^\mu,u^\nu]_+]_+=-[u_\mu,[u_\nu,[u^\mu,u^\nu]_+]_+]_+\\
&[u_\mu,[u_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]]-[[u_\mu,u_\nu]_+,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]_+=-[u_\mu,[u_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]_+]_+. \qquad \text{(A2)}
\end{aligned}
$$

The reason for considering (A1) and (A2) separately is because the terms in them do not mix. One thus can take $i = 12, 13, 26, 27, 29$ and $i = 46, 47, 48$ of Table 1 as the two sets of linearly independent terms.

B

In this appendix, we show how to obtain a set of linearly independent terms from (19).

Using (9), (18),(20)- (23), one gets the following 15 identities in the following 22 terms:

$$
\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+]-\frac{1}{4}[[u_\mu,u_\nu],[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+=-[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]]_+]_+-\frac{1}{16}[[u_\mu,u_\nu],[u_\lambda,\mathrm{S}\cdot u]]=-\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]]_+]_+\Big)
\end{aligned}
$$

$$
\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]]]-\frac{1}{16}[[u_\mu,u_\nu],[u_\lambda,\mathrm{S}\cdot u]]=-\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([u_\mu,[u_\nu,[u_\lambda,\mathrm{S}\cdot u]_+]_+]-[[u_\mu,u_\nu],[u_\lambda,\mathrm{S}\cdot u]_+]_+=-[u_\mu,[u_\nu,[u_\lambda,\mathrm{S}\cdot u]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([u_\mu,[u_\nu,[u_\lambda,\mathrm{S}\cdot u]]_+]_+-[[u_\mu,u_\nu],[u_\lambda,\mathrm{S}\cdot u]]=-[u_\mu,[u_\nu,[u_\lambda,\mathrm{S}\cdot u]]_+]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([u_\mu,[u_\nu,[u_\lambda,\mathrm{S}\cdot u]]]-[[u_\mu,u_\nu],[u_\lambda,\mathrm{S}\cdot u]]=-[u_\mu,[u_\nu,[u_\lambda,\mathrm{S}\cdot u]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(\frac{1}{4}[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,[u_\nu,u_\lambda]]_+]_+-\frac{1}{16}[[\mathrm{S}\cdot u,u_\mu],[u_\nu,u_\lambda]]=\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[u_\nu,u_\lambda]]_+]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(\frac{1}{4}[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,[u_\nu,u_\lambda]]_+]_+-\frac{1}{16}[[\mathrm{S}\cdot\mathrm{D},\mathrm{D}_\mu]_+,[u_\nu,u_\lambda]]_+=-\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[u_\nu,u_\lambda]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(\frac{1}{4}[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,[u_\nu,u_\lambda]]]-\frac{1}{16}[[\mathrm{S}\cdot u,u_\mu],[u_\nu,u_\lambda]]=\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[u_\nu,u_\lambda]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([\mathrm{S}\cdot u,[u_\mu,[u_\nu,u_\lambda]]_+]_+-[[\mathrm{S}\cdot u,u_\mu],[u_\nu,u_\lambda]]=[u_\mu,[\mathrm{S}\cdot u,[u_\nu,u_\lambda]]_+]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([\mathrm{S}\cdot u,[u_\mu,[u_\nu,u_\lambda]]_+]_+-[[\mathrm{S}\cdot u,u_\mu]_+,[u_\nu,u_\lambda]_+]=-[u_\mu,[\mathrm{S}\cdot u,[u_\nu,u_\lambda]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([\mathrm{S}\cdot u,[u_\mu,[u_\nu,u_\lambda]]]-[[\mathrm{S}\cdot u,u_\mu],[u_\nu,u_\lambda]]=[u_\mu,[\mathrm{S}\cdot u,[u_\nu,u_\lambda]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([u_\mu,[u_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+]-[[u_\mu,u_\nu],[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+=-[u_\mu,[u_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([u_\mu,[u_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+]-[[u_\mu,u_\nu],[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+=-[u_\mu,[u_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]_+]_+]-[[u_\mu,u_\nu],[u_\lambda,\mathrm{S}\cdot\mathrm{D}]_+]_+=-[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big([u_\mu,[\mathrm{S}\cdot u,[u_\nu,u_\lambda]]_+]_+-\frac{1}{4}[[u_\mu,\mathrm{S}\cdot u]_+,[u_\nu,u_\lambda]]_+=-\frac{1}{4}[\mathrm{S}\cdot u,[u_\mu,[u_\nu,u_\lambda]]]\Big).
\end{aligned}
\tag{B1}
$$

One can thus take $i=14,15,30,31,49,50,51$ of Table 1 as linearly independent terms.

C

In this appendix, we show how to obtain a set of linearly independent terms from (24).

Using (9) and (24), one gets 24 identities in 30 terms:

$$
\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\kappa,[\mathrm{D}_\mu,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+]-[[\mathrm{D}_\kappa,\mathrm{D}_\mu]_+,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]=-[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\kappa,[\mathrm{D}_\mu,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+]-\frac{1}{4}[[u_\kappa,u_\mu],[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+=-[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\kappa,[\mathrm{D}_\mu,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+-[[\mathrm{D}_\kappa,\mathrm{D}_\mu]_+,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]=-[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big(\frac{1}{4}[\mathrm{D}_\kappa,[\mathrm{D}_\mu,[u^\kappa,u_\nu]]_+]_+-\frac{1}{16}[[u_\kappa,u_\mu],[u^\kappa,u_\nu]]=-\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u^\kappa,u_\nu]]_+]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big(\frac{1}{4}[\mathrm{D}_\kappa,[\mathrm{D}_\mu,[u^\kappa,u_\nu]]_+]_+-\frac{1}{4}[[u_\kappa,u_\mu],[\mathrm{D}^\kappa,u_\nu]_+]_+=-\frac{1}{4}[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u^\kappa,u_\nu]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[u^\kappa,u_\nu]_+]_+]-[[u^\kappa,u_\mu]_+,[u_\kappa,u_\nu]_+]=-[u_\mu,[u_\kappa,[u^\kappa,u_\nu]_+]_+]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[u^\kappa,u_\nu]_+]_+]-[[u^\kappa,u_\mu],[u_\kappa,u_\nu]_+]_+=-[u_\mu,[u_\kappa,[u^\kappa,u_\nu]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[u^\kappa,u_\nu]_+]]_+-[[u^\kappa,u_\mu]_+,[u_\kappa,u_\nu]_+]=-[u_\mu,[u_\kappa,[u^\kappa,u_\nu]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[u^\kappa,u_\nu]]_+]_+-[[u^\kappa,u_\mu],[u_\kappa,u_\nu]]=-[u_\mu,[u_\kappa,[u^\kappa,u_\nu]]_+]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[u^\kappa,u_\nu]]_+]_+-[[u^\kappa,u_\mu]_+,[u_\kappa,u_\nu]]_+=-[u_\mu,[u_\kappa,[u^\kappa,u_\nu]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[u^\kappa,u_\nu]]]-[[u^\kappa,u_\mu],[u_\kappa,u_\nu]]=-[u_\mu,[u_\kappa,[u^\kappa,u_\nu]]]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+]-[[u_\kappa,u_\mu]_+,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]=-[u_\mu,[u_\kappa,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+-[[u_\kappa,u_\mu],[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]_+=-[u_\mu,[u_\kappa,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([u_\kappa,[u_\mu,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+-[[u_\kappa,u_\mu]_+,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]=-[u_\mu,[u_\kappa,[\mathrm{D}^\kappa,\mathrm{D}_\nu]_+]]_+\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\kappa,[\mathrm{D}_\mu,[u^\kappa,u_\nu]_+]_+]-[[\mathrm{D}_\kappa,\mathrm{D}_\mu]_+,[u^\kappa,u_\nu]_+]=-[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u^\kappa,u_\nu]_+]_+]\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\kappa,[\mathrm{D}_\mu,[u^\kappa,u_\nu]_+]_+]-\frac{1}{4}[[u_\kappa,u_\mu],[u^\kappa,u_\nu]_+]_+=-[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u^\kappa,u_\nu]_+]]_+\Big)\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\kappa,[\mathrm{D}_\mu,[u^\kappa,u_\nu]_+]]_+-[[\mathrm{D}_\kappa,\mathrm{D}_\mu]_+,[u^\kappa,u_\nu]_+]=-[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u^\kappa,u_\nu]_+]]_+\Big).
\end{aligned}
\tag{C1}
$$

One can thus take $i=7,22,23,34,55,56$ as linearly independent terms.

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Table 1: The Allowed $\mathrm{O}(q^4, \phi^{2n})$ Terms

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)[d_i]	E ON
1	(4,0,0,0)	$(v\cdot \mathrm{D})^4$	D[d_{197}]	E
2		$\mathrm{D}_\mu(v\cdot \mathrm{D})^2\mathrm{D}^\mu$	D[d_{198}]	E
3		$[\mathrm{D}^2,(v\cdot \mathrm{D})^2]_+$	F	E
4		$v\cdot \mathrm{D}\mathrm{D}_\mu v\cdot \mathrm{D}\mathrm{D}^\mu + \mathrm{h.c.}$	F	E
5		D^4	F	E
6		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}^2,[\mathrm{D}_\mu,\mathrm{D}_\nu]]_+$	F	E
7		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\kappa[\mathrm{D}_\mu,\mathrm{D}_\nu]\mathrm{D}^\kappa$	D[d_{178}]	E
8		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\mu \mathrm{D}^2\mathrm{D}_\nu$	F	E
9		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[(v\cdot \mathrm{D})^2,[\mathrm{D}_\mu,\mathrm{D}_\nu]]_+$	F	E
10		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{v}\cdot \mathrm{D}[\mathrm{D}_\mu,\mathrm{D}_\nu]\mathrm{v}\cdot \mathrm{D}$	D[d_{177}]	E
11		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\mu(\mathrm{v}\cdot \mathrm{D})^2\mathrm{D}_\nu$	F	E
12		$[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]]$	F	E
13		$[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]_+]_+$	F	E
14		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D}]_+]]_+$	F	E
15		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,\mathrm{S}\cdot \mathrm{D}]_+]_+]$	F	E
16		$\epsilon^{\mu\nu\rho\lambda}S_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,v\cdot \mathrm{D}]_+]_+]$	F	E
17		$\epsilon^{\mu\nu\rho\lambda}S_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,v\cdot \mathrm{D}]_+]]_+$	F	E
18	(0,4,0,0)	$(v\cdot u)^4$	D[d_5]	ON
19		$[u^2,(v\cdot u)^2]_+$	D[d_3]	ON
20		$v\cdot uu_\mu v\cdot uu^\mu + \mathrm{h.c.}$	E	ON
21		u^4	D[d_2]	ON
22		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u^2,[u_\mu,u_\nu]]_+$	D[d_{31}]	ON
23		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{u}_\kappa[\mathrm{u}_\mu,\mathrm{u}_\nu]\mathrm{u}^\kappa$	D[d_{32}]	ON
24		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[(v\cdot u)^2,[u_\mu,u_\nu]]_+$	D[d_{34}]	ON
25		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{v}\cdot \mathrm{u}[\mathrm{u}_\mu,\mathrm{u}_\nu]\mathrm{v}\cdot \mathrm{u}$	D[d_{35}]	ON
26		$[u_\mu,u_\nu]^2$	F	ON
27		$[u_\mu,u_\nu]_+^2$	D[d_1]	ON
28		$[[v\cdot u,u^\mu],[v\cdot u,u_\mu]_+]$	F	ON
29		$[u_\mu,[u_\nu.[u^\mu,u^\nu]_+]]$	F	ON
30		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{S}\cdot u,[u_\lambda,[u_\mu,u_\nu]]_+]_+$	D[d_{24}]	ON
31		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\lambda,[\mathrm{S}\cdot u,[u_\mu,u_\nu]]_+]_+$	D[d_{25}]	ON
32		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[v\cdot u,[u_\lambda,[u_\mu,u_\nu]]_+]_+$	E	ON
33		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[u_\lambda,[v\cdot u,[u_\mu,u_\nu]]_+]_+$	E	ON

Table 1: continued

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)$[d_i]$	E ON
34		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[u_\mu,u_\kappa],[u_\nu,u^\kappa]_+]_+$	D$[d_{67}]$	ON
35		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[u_\mu,v\cdot u],[u_\nu,v\cdot u]_+]_+$	D$[d_{36}]$	ON
36		$[u_\mu,v\cdot u]_+^2$	D$[d_4]$	ON
37		$u_\mu(v\cdot u)^2u^\mu$	F	ON
38	(2,2,0,0)	$v\cdot \mathrm{D}(v\cdot u)^2 v\cdot \mathrm{D}$	D$[d_{146}]$	E
39		$[v\cdot \mathrm{D},(v\cdot u)]^2$	D$[d_{128}]$	ON
40		$v\cdot \mathrm{D}u^2v\cdot \mathrm{D}$	D$[d_{149}]$	E
41		$[v\cdot \mathrm{D},u_\mu]^2$	D$[d_{133}]$	ON
42		$\mathrm{D}_\mu(v\cdot u)^2\mathrm{D}^\mu$	D$[d_{147}]$	ON
43		$[\mathrm{D}_\mu,(v\cdot u)]^2$	D$[d_{135}]$	ON
44		$v\cdot \mathrm{D}[v\cdot u,u^\mu]_+\mathrm{D}_\mu+\mathrm{h.c.}$	D$[d_{148}]]$	E
45		$[\mathrm{D}_\mu,[v\cdot \mathrm{D},[v\cdot u,u^\mu]_+]]$	F	E
46		$[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u^\mu,u^\nu]_+]]$	F	E
47		$[u_\mu,[u_\nu,[\mathrm{D}^\mu,\mathrm{D}^\nu]_+]]$	F	E
48		$[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u^\mu,u^\nu]_+]_+]_+$	F	E
49		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{S}\cdot \mathrm{D},[\mathrm{D}_\mu,[u_\nu,u_\lambda]]_+]_+$	F	E
50		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]_+]]_+$	F	E
51		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,\mathrm{S}\cdot u]_+]_+]$	F	E
52		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[v\cdot \mathrm{D},[\mathrm{D}_\mu,[u_\nu,u_\lambda]]_+]_+$	F	E
53		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,v\cdot u]_+]]_+$	F	E
54		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[u_\lambda,v\cdot u]_+]_+]$	F	E
55		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u_\nu,u^\kappa]]]$	D$[d_{179}]$	E
56		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[\mathrm{D}_\kappa,[u_\nu,u^\kappa]]_+]_+$	F	E
57		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda v\cdot \mathrm{D}[v\cdot u,u_\mu]\mathrm{D}_\nu+\mathrm{h.c.}$	D$[d_{176}]$	E
58		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[v\cdot \mathrm{D},[u_\nu,v\cdot u]]]$	F	E
59		$[\mathrm{D}_\mu,u_\nu]^2$	D$[d_{136}]$	ON
60		$\mathrm{D}_\mu u^2\mathrm{D}^\mu$	D$[d_{150}]$	E
61		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[\mathrm{D}_\nu,u^2]]_+$	F	E
62		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[\mathrm{D}_\nu,(v\cdot u)^2]]_+$	F	E
63		$[v\cdot u,[[v\cdot \mathrm{D},v\cdot u],v\cdot \mathrm{D}]]_+$	D$[d_{129}]$	ON
64		$[v\cdot \mathrm{D},[[v\cdot \mathrm{D},v\cdot u],v\cdot u]]_+$	D$[d_{138}]$	E
65		$[u_\mu,[[v\cdot \mathrm{D},u^\mu],v\cdot \mathrm{D}]]_+$	D$[d_{134}]$	ON

Table 1: continued

i	(m,n,p,q)	Terms	F($\equiv$Finite) D($\equiv$Divergent)$[d_i]$	E ON
66		$[v\cdot \mathrm{D},[[v\cdot \mathrm{D},u_\mu],u^\mu]]_+$	D$[d_{142}]$	E
67		$[v\cdot u,[[\mathrm{D}^\mu,v\cdot u],\mathrm{D}_\mu]]_+$	D$[d_{131}]$	ON
68		$[\mathrm{D}_\mu,[[\mathrm{D}^\mu,v\cdot u],v\cdot u]]_+$	D$[d_{141}]$	ON
69		$[u^\nu,[[\mathrm{D}_\mu,u_\nu]_+,\mathrm{D}^\mu]]$	D$[d_{137}]$	E
70		$[\mathrm{D}_\mu,[[\mathrm{D}^\mu,u_\nu],u^\nu]]_+$	D$[d_{143}]$	E
71		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\mu],u^\nu]_+]$	F	E
72		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\mu]_+,u^\nu]_+]_+$	F	E
73		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\mu],u^\nu]]_+$	D$[d_{145}]$	E
74		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\mu]_+,u^\nu]]$	F	E
75		$[u_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,u^\nu]]_+]$	F	ON
76		$[u_\mu,[\mathrm{D}_\nu,[\mathrm{D}^\mu,u^\nu]]]_+$	F	ON
77		$[u_\mu,[[\mathrm{D}^\mu,v\cdot u],v\cdot \mathrm{D}]]_+$	D$[d_{130}]$	E
78		$[v\cdot \mathrm{D},[[\mathrm{D}_\nu,v\cdot u],u^\nu]]_+$	D$[d_{144}]$	E
79		$[[\mathrm{D}_\mu,v\cdot u],[v\cdot \mathrm{D},u^\mu]]_+$	D$[d_{132}]$	ON
80		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u^\mu],v\cdot u]]_+$	D$[d_{140}]$	E
81		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u^\mu]_+,v\cdot u]]$	F	E
82		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u^\mu]_+,v\cdot u]_+]_+$	F	E
83		$[v\cdot \mathrm{D},[u^\mu,[\mathrm{D}_\mu,v\cdot u]]_+]$	F	E
84		$[v\cdot \mathrm{D},[u^\mu,[\mathrm{D}_\mu,v\cdot u]]]_+$	F	E
85		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\nu]_+,u^\mu]]$	F	E
86		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\nu],u^\mu]]_+$	D$[d_{123}]$	E
87		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\nu]_+,u^\mu]_+]_+$	F	E
88		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\nu],u^\mu]_+]$	F	E
89		$[u_\mu,[[\mathrm{D}_\nu,u^\nu],\mathrm{D}^\mu]_+]$	F	ON
90		$[u_\mu,[[\mathrm{D}_\nu,u^\nu],\mathrm{D}^\mu]]_+$	F	ON
91		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},v\cdot u]_+,u^\mu]]$	F	ON
92		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},v\cdot u],u^\mu]]_+$	D$[d_{139}]$	ON
93		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},v\cdot u]_+,u^\mu]_+]_+$	F	ON
94		$[\mathrm{D}_\mu,[[v\cdot \mathrm{D},v\cdot u],u^\mu]_+]$	F	ON
95		$[u_\mu,[[v\cdot \mathrm{D},v\cdot u],\mathrm{D}^\mu]_+]$	F	ON
96		$[u_\mu,[[v\cdot \mathrm{D},v\cdot u],\mathrm{D}^\mu]]_+$	F	ON
97		$[v\cdot \mathrm{D},[[u_\mu,\mathrm{D}^\mu]_+,v\cdot u]_+]_+$	F	E
98		$[v\cdot \mathrm{D},[[u_\mu,\mathrm{D}^\mu],v\cdot u]]_+$	F	E

Table 1: continued

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)$[d_i]$	E ON
99		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda],\mathrm{S}\cdot u]]$	F	E
100		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,\mathrm{S}\cdot u]_+]$	F	E
101		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,\mathrm{S}\cdot u]]_+$	F	E
102		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda],\mathrm{S}\cdot u]_+]_+$	F	E
103		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,\mathrm{S}\cdot u],u_\lambda]_+]_+$	F	E
104		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,\mathrm{S}\cdot u]_+,u_\lambda]_+]$	F	E
105		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,\mathrm{S}\cdot u]_+,u_\lambda]_+]$	F	E
106		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,\mathrm{S}\cdot u],u_\lambda]]$	F	E
107		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda],v\cdot u]]$	F	E
108		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,v\cdot u]_+]$	F	E
109		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda]_+,v\cdot u]]_+$	F	E
110		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u_\lambda],v\cdot u]_+]_+$	F	E
111		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,v\cdot u]_+,u_\lambda]]_+$	F	E
112		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,v\cdot u]_+,u_\lambda]_+]$	F	E
113		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,v\cdot u],u_\lambda]]$	F	E
114		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\nu,[[\mathrm{D}_\mu,v\cdot u],u_\lambda]_+]_+$	F	E
115		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{S}\cdot\mathrm{D},u_\nu],u_\lambda]]$	F	E
116		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{S}\cdot\mathrm{D},u_\nu]_+,u_\lambda]_+]$	F	E
117		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{S}\cdot\mathrm{D},u_\nu],u_\lambda]_+]_+$	F	E
118		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[[\mathrm{S}\cdot\mathrm{D},u_\nu]_+,u_\lambda]]_+$	F	E
119		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[v\cdot\mathrm{D},u_\nu],u_\lambda]]$	F	E
120		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{S}\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda]_+,u_\nu]]_+$	F	E
121		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{S}\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda]_+,u_\nu]]_+$	F	E
122		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{S}\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda]_+,u_\nu]_+]$	F	E
123		$i\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{S}\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda],u_\nu]]$	F	E
124		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[v\cdot\mathrm{D},u_\nu]_+,u_\lambda]]_+$	F	E
125		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[\mathrm{D}_\mu,[[v\cdot\mathrm{D},u_\nu],u_\lambda]_+]_+$	F	E
126		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[[\mathrm{D}_\mu,u_\lambda],[v\cdot\mathrm{D},u_\nu]_+]_+$	D$[d_{188}]$	E
127		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda],u_\nu]_+]_+$	F	E
128		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda]_+,u_\nu]]_+$	F	E
129		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda]_+,u_\nu]]_+$	F	E
130		$i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\rho[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\lambda],u_\nu]]$	F	E
131		$i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\kappa],u_\kappa]]$	F	E

Table 1: continued

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)[d_i]	E ON
132		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\kappa]_+,u_\kappa]_+]$	F	E
133		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\kappa],u_\kappa]_+]_+$	F	E
134		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,u^\kappa]_+,u_\kappa]]_+$	F	E
135		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\kappa,[[\mathrm{D}_\nu,u^\kappa]_+,\mathrm{D}_\mu]]_+$	F	E
136		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\kappa,[[\mathrm{D}_\nu,u^\kappa]_+,\mathrm{D}_\mu]_+]$	F	E
137		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,v\cdot u],v\cdot u]]$	F	E
138		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,v\cdot u]_+,v\cdot u]_+]$	F	E
139		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,v\cdot u],v\cdot u]_+]_+$	F	E
140		$i\epsilon^{\mu\nu\rho\lambda}v_\mu \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\nu,v\cdot u]_+,v\cdot u]]_+$	F	E
141		$i\epsilon^{\mu\nu\rho\lambda}v_\mu \mathrm{S}_\lambda[v\cdot u,[[\mathrm{D}_\nu,v\cdot u]_+,u_\mu]]_+$	F	E
142		$i\epsilon^{\mu\nu\rho\lambda}v_\mu \mathrm{S}_\lambda[v\cdot u,[[\mathrm{D}_\nu,v\cdot u]_+,u_\mu]_+]$	F	E
143		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u_\nu],u^\kappa]]$	F	E
144		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u_\nu]_+,u^\kappa]_+]$	F	E
145		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u_\nu],u^\kappa]_+]_+$	F	E
146		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u_\nu]_+,u^\kappa]]_+$	F	E
147		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\kappa],u^\nu]_+]_+$	F	E
148		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\kappa]_+,u^\nu]]_+$	F	E
149		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\kappa]_+,u^\nu]_+]$	F	E
150		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\kappa],u^\nu]]$	F	E
151		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u_\nu],v\cdot u]]$	D[d_{166}]	E
152		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[\mathrm{D}_\mu,v\cdot u],[v\cdot \mathrm{D},u_\nu]]$	D[d_{163}]	ON
153		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u_\nu],v\cdot u]_+]_+$	D[d_{172}]	E
154		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot \mathrm{D},[[\mathrm{D}_\mu,v\cdot u],u_\nu]]$	Dd_{164}]	E
155		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u_\nu]_+,v\cdot u]]_+$	F	E
156		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot \mathrm{D},u_\nu]_+,v\cdot u]_+]$	F	E
157		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot \mathrm{D},[[\mathrm{D}_\mu,v\cdot u]_+,u_\nu]_+]$	F	E
158		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot \mathrm{D},[[\mathrm{D}_\mu,v\cdot u]_+,u_\nu]]_+$	F	E
159		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\nu],u^\kappa]]$	F	E
160		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\nu]_+,u^\kappa]_+]$	F	E
161		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\nu],u^\kappa]_+]_+$	F	E
162		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}_\mu,u_\nu]_+,u^\kappa]]_+$	F	E

Table 1: continued

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)$[d_i]$	E 0N
163		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u^\kappa],u_\nu]]$	F	E
164		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u^\kappa]_+,u_\nu]_+]$	F	E
165		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u^\kappa]_+,u_\nu]]_+$	F	E
166		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[\mathrm{D}_\kappa,u^\kappa],u_\nu]_+]_+$	F	E
167		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\nu],v\cdot u]]$	$\mathrm{D}[d_{165}]$	E
168		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[v\cdot\mathrm{D},v\cdot u],[\mathrm{D}_\mu,u_\nu]]$	$\mathrm{D}[d_{162}]$	ON
169		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\nu],v\cdot u]_+]_+$	$\mathrm{D}[d_{173}]$	E
170		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\nu]_+,v\cdot u]]_+$	F	E
171		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[\mathrm{D}_\mu,u_\nu]_+,v\cdot u]_+]$	F	E
172		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot\mathrm{D},v\cdot u],u_\nu]]$	$\mathrm{D}[d_{171}]$	E
173		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot\mathrm{D},v\cdot u]_+,u_\nu]]_+$	F	E
174		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[[v\cdot\mathrm{D},v\cdot u]_+,u_\nu]_+]$	F	E
175		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\mu,[[\mathrm{D}^\kappa,u_\nu],\mathrm{D}_\kappa]]$	$\mathrm{D}[d_{170}]$	ON
176		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[\mathrm{D}_\kappa,u_\mu],[\mathrm{D}^\kappa,u_\nu]]$	$\mathrm{D}[d_{169}]$	ON
177		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}^\kappa,u_\mu],u_\nu]_+]_+$	$\mathrm{D}[d_{175}]$	E
178		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}^\kappa,u_\mu]_+,u_\nu]]_+$	F	E
179		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\mu,[[\mathrm{D}^\kappa,u_\nu]_+,\mathrm{D}_\kappa]]_+$	F	E
180		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}^\kappa,u_\mu]_+,u_\nu]_+]$	F	E
181		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\mu,[[v\cdot\mathrm{D},u_\nu],v\cdot\mathrm{D}]]$	$\mathrm{D}[d_{168}]$	ON
182		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[v\cdot\mathrm{D},u_\mu],[v\cdot\mathrm{D},u_\mu]]$	$\mathrm{D}[d_{167}]$	ON
183		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[v\cdot\mathrm{D},u_\mu],u_\nu]_+]_+$	$\mathrm{D}[d_{174}]$	E
184		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[v\cdot\mathrm{D},u_\mu]_+,u_\nu]]_+$	F	E
185		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[u_\mu,[[v\cdot\mathrm{D},u_\nu]_+,v\cdot\mathrm{D}]_+]$	F	E
186		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[v\cdot\mathrm{D},[[v\cdot\mathrm{D},u_\mu]_+,u_\nu]_+]$	F	E

Table 1: continued

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)[d_i]	E ON
187	(2,0,1,0)	$[\mathrm{D}_\mu,[\mathrm{D}^\mu,\chi_+]]$	D[d_{158}]	E
188		$\mathrm{D}_\mu\chi_+\mathrm{D}^\mu$	D[d_{160}]	E
189		$v\cdot\mathrm{D}\chi_+\mathrm{v}\cdot\mathrm{D}$	D[d_{159}]	E
190		$[v\cdot\mathrm{D},[\mathrm{v}\cdot\mathrm{D},\chi_+]]$	D[d_{157}]	E
191		$i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\mu,[\mathrm{D}_\nu,\chi_+]_+]$	F	E
192	(1,1,0,1)	$i[v\cdot\mathrm{D},[v\cdot u,\chi_-]]_+$	F	E
193		$i[v\cdot\mathrm{D},[v\cdot u,\chi_-]_+]$	F	E
194		$i[v\cdot u,[v\cdot\mathrm{D},\chi_-]]_+$	F	ON
195		$i[\mathrm{D}_\mu,[u^\mu,\chi_-]]_+$	F	E
196		$i[u_\mu,[\mathrm{D}^\mu,\chi_-]]_+$	F	ON
197		$i[\mathrm{D}_\mu,[u^\mu,\chi_-]_+]$	F	E
198		$\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\mu,[u_\nu,\chi_-]],$	F	E
199		$\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[u_\mu,[\mathrm{D}_\nu,\chi_-]],$	F	ON
200		$\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\mu,[u_\nu,\chi_-]_+]_+$	F	E
201	(0,2,1,0)	$u_\mu\chi_+u^\mu$	F	ON
202		$u^2\chi_+$	D[d_{10}]	ON
203		$v\cdot u\chi_+v\cdot u$	F	ON
204		$(v\cdot u)^2\chi_+$	D[d_{13}]	ON
205		$i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[[u_\mu,u_\nu],\chi_+]_+$	D[d_{51}]	ON
206	(0,0,2,0)	χ_+^2	D[d_{21}]	ON
207	(0,0,0,2)	χ_-^2	F	ON

Table 2: The Allowed $\mathrm{O}(q^4, \phi^{2n+1})$ Terms

i	(m,n,p,q)	Terms	F($\equiv$Finite) D($\equiv$Divergent)[d_i]	E ON
208	(3,1,0,0)	$i[\mathrm{D}_\mu,[v\cdot\mathrm{D},[\mathrm{D}^\mu,\mathrm{S}\cdot u]]]_+$	D[d_{191}]	E
209		$i[v\cdot\mathrm{D},[\mathrm{D}_\mu,[\mathrm{D}^\mu,\mathrm{S}\cdot u]_+]]$	F	E
210		$i[\mathrm{D}_\mu,[\mathrm{D}^\mu,[v\cdot\mathrm{D},\mathrm{S}\cdot u]]]_+$	F	E
211		$i[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},[\mathrm{D}_\mu,u^\mu]]]_+$	F	ON$'$
212		$i[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,u^\mu]_+]]$	F	E
213		$i[\mathrm{D}_\mu,[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},u^\mu]]]_+$	F	E
214		$i[v\cdot\mathrm{D},[\mathrm{D}_\mu[\mathrm{S}\cdot\mathrm{D},u^\mu]_+]]$	F	E
215		$i[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},u^\mu]_+]_+]_+$	F	E
216		$i[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},u^\mu]_+]]$	F	E
217		$i[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}^\mu,v\cdot u]_+]]$	F	E
218		$i[\mathrm{D}_\mu,[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}^\mu,v\cdot u]_+]_+]_+$	F	E
219		$i[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,[v\cdot\mathrm{D},u^\mu]]]_+$	F	E
220		$i[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,[\mathrm{D}^\mu,v\cdot u]]]_+$	F	E
221		$i[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\mu,u^\mu]_+]_+]_+$	F	E
222		$i[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},[v\cdot\mathrm{D},v\cdot u]]]_+$	D[d_{189}]	E
223		$i[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},v\cdot u]_+]]$	F	ON
224		$i[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},v\cdot u]_+]_+]_+$	F	ON
225		$i[v\cdot\mathrm{D},[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},v\cdot u]_+]]$	D[d_{194}]	E
226		$i\mathrm{D}_\mu[v\cdot\mathrm{D},\mathrm{S}\cdot u]_+\mathrm{D}^\mu$	D[d_{196}]	E
227		$i[\mathrm{D}_\mu,[\mathrm{D}^\mu,[\mathrm{S}\cdot\mathrm{D},v\cdot u]]]_+$	F	E
228		$i[v\cdot\mathrm{D},[v\cdot\mathrm{D},[v\cdot\mathrm{D},\mathrm{S}\cdot u]]]_+$	D[d_{190}]	E
229		$i[\mathrm{S}\cdot u,(v\cdot\mathrm{D})^3]_+$	D[d_{193}]	ON$'$
230		$iv\cdot\mathrm{D}[v\cdot\mathrm{D},\mathrm{S}\cdot u]_+v\cdot\mathrm{D}$	D[d_{195}]	E
231		$\epsilon^{\mu\nu\rho\lambda}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\rho,u_\lambda]_+]_+]$	F	E
232		$\epsilon^{\mu\nu\rho\lambda}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\rho,u_\lambda]]_+]_+$	F	E
233		$\epsilon^{\mu\nu\rho\lambda}[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\rho,u_\lambda]_+]]_+$	F	E
234		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[v\cdot\mathrm{D},u_\lambda]]_+]_+$	D[d_{72}]	ON
235		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[v\cdot\mathrm{D},u_\lambda]_+]]_+$	F	E
236		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[v\cdot\mathrm{D},u_\lambda]_+]_+]$	F	E
237		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,v\cdot u]]_+]_+$	F	ON
238		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,v\cdot u]_+]]_+$	F	ON
239		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[\mathrm{D}_\nu,[\mathrm{D}_\lambda,v\cdot u]_+]_+]$	F	ON
240		$\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot\mathrm{D},[\mathrm{D}_\mu,[\mathrm{D}_\nu,u_\lambda]_+]]_+$	F	E
241		$\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot\mathrm{D},[\mathrm{D}_\mu,[\mathrm{D}_\nu,u_\lambda]_+]_+]$	F	E
242		$\epsilon^{\mu\nu\rho\lambda}v_\rho[[v\cdot\mathrm{D},\mathrm{D}_\mu]_+,[\mathrm{D}_\nu,u_\lambda]_+]$	F	E

Table 2: continued

i	(m,n,p,q)	Terms	F($\equiv$Finite) D($\equiv$Divergent)[d_i]	E ON
243		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[\mathrm{D}_\nu,[v\cdot\mathrm{D},\mathrm{D}_\lambda]_+]_+]$	F	E
244	(1,3,0,0)	$i[v\cdot u,u_\mu]_+,[u^\mu,\mathrm{S}\cdot\mathrm{D}]_+]_+$	D[d_{84}]	ON$'$
245		$i[[\mathrm{D}_\mu,\mathrm{S}\cdot u],[v\cdot u,u^\mu]]_+$	D[d_{81}]	ON
246		$i[[\mathrm{D}_\mu,\mathrm{S}\cdot u],[v\cdot u,u^\mu]_+]$	F	ON
247		$i[\mathrm{S}\cdot u,[\mathrm{D}_\mu,[u^\mu,v\cdot u]]]_+$	F	ON
248		$i[\mathrm{D}_\mu,[\mathrm{S}\cdot u,[u^\mu,v\cdot u]]]_+$	F	E
249		$i[v\cdot u,[u_\mu,[\mathrm{D}^\mu,\mathrm{S}\cdot u]]_+]$	F	ON
250		$i[u_\mu,[v\cdot u,[\mathrm{D}^\mu,\mathrm{S}\cdot u]]_+]$	F	ON
251		$i[v\cdot u,[u_\mu,[\mathrm{D}^\mu,\mathrm{S}\cdot u]]]_+$	F	ON
252		$i[u_\mu,[v\cdot u,[\mathrm{D}^\mu,\mathrm{S}\cdot u]]]_+$	F	ON
253		$i[[v\cdot u,\mathrm{S}\cdot u],[\mathrm{D}_\mu,u^\mu]_+]$	F	ON$'$
254		$i[u_\mu,[\mathrm{D}^\mu,[\mathrm{S}\cdot u,v\cdot u]]]_+$	F	ON
255		$i[\mathrm{D}_\mu,[u^\mu,[\mathrm{S}\cdot u,v\cdot u]]]_+$	E	ON
256		$i[\mathrm{S}\cdot u,[v\cdot u,[\mathrm{D}^\mu,u_\mu]]_+]$	F	ON
257		$i[\mathrm{S}\cdot u,[v\cdot u,[\mathrm{D}^\mu,u_\mu]]]_+$	F	ON
258		$i[v\cdot u,[\mathrm{S}\cdot u,[\mathrm{D}^\mu,u_\mu]]_+]$	F	ON
259		$i[v\cdot u,[\mathrm{S}\cdot u,[\mathrm{D}^\mu,u_\mu]]]_+$	F	ON
260		$i[\mathrm{D}_\mu,[u^\mu,[v\cdot u,\mathrm{S}\cdot u]_+]]$	F	E
261		$i[[v\cdot u,u_\mu],[\mathrm{S}\cdot\mathrm{D},u^\mu]_+]$	F	ON$'$
262		$i[u^\mu,[\mathrm{S}\cdot\mathrm{D},[u_\mu,v\cdot u]]]_+$	F	ON
263		$i[[\mathrm{S}\cdot\mathrm{D},[u^\mu,[u_\mu,v\cdot u]]]_+$	F	E
264		$i[v\cdot u,[u^\mu,[\mathrm{S}\cdot\mathrm{D},u_\mu]]_+]$	F	ON
265		$i[v\cdot u,[u^\mu,[\mathrm{S}\cdot\mathrm{D},u_\mu]]]_+$	F	ON
266		$i[u^\mu,[v\cdot u,[\mathrm{S}\cdot\mathrm{D},u_\mu]]_+]$	F	ON
267		$i[u^\mu,[v\cdot u,[\mathrm{S}\cdot\mathrm{D},u_\mu]]]_+$	F	ON
268		$i[u^2,[v\cdot u,\mathrm{S}\cdot\mathrm{D}]_+]_+$	D[d_{83}]	ON$'$
269		$i[[u^\mu,\mathrm{S}\cdot u],[v\cdot\mathrm{D},u_\mu]_+]$	F	ON$'$
270		$i[v\cdot\mathrm{D},[u_\mu,[\mathrm{S}\cdot u,u^\mu]]]_+$	F	ON
271		$i[u_\mu,[v\cdot\mathrm{D},[\mathrm{S}\cdot u,u^\mu]]]_+$	F	ON
272		$i[\mathrm{S}\cdot u,[u_\mu,[v\cdot\mathrm{D},u^\mu]]_+]$	F	ON
273		$i[\mathrm{S}\cdot u,[u_\mu,[v\cdot\mathrm{D},u^\mu]]]_+$	F	ON
274		$i[u_\mu,[\mathrm{S}\cdot u,[v\cdot\mathrm{D},u^\mu]]_+]$	F	ON
275		$i[u_\mu,[\mathrm{S}\cdot u,[v\cdot\mathrm{D},u^\mu]]]_+$	F	ON

Table 2: continued

i	(m,n,p,q)	Terms	F($\equiv$Finite) D($\equiv$Divergent)$[d_i]$	E ON
276		$i[u^2,[\mathrm{S}\cdot u, v\cdot\mathrm{D}]_+]_+$	$\mathrm{D}[d_{85}]$	ON$'$
277		$i[[v\cdot u,\mathrm{S}\cdot u],[v\cdot\mathrm{D},v\cdot u]]_+$	$\mathrm{D}[d_{82}]$	ON$'$
278		$i[[v\cdot u,\mathrm{S}\cdot u]_+,[v\cdot\mathrm{D},v\cdot u]_+]_+$	F	ON$'$
279		$i[v\cdot\mathrm{D},[v\cdot u,[\mathrm{S}\cdot u,v\cdot u]]]_+$	F	E
280		$i[v\cdot u,[v\cdot\mathrm{D},[\mathrm{S}\cdot u,v\cdot u]]]_+$	F	ON
281		$i[\mathrm{S}\cdot u,[v\cdot u,[v\cdot\mathrm{D},v\cdot u]]_+]$	F	ON
282		$i[\mathrm{S}\cdot u,[v\cdot u,[v\cdot\mathrm{D},v\cdot u]]]_+$	F	ON
283		$i[v\cdot u,[\mathrm{S}\cdot u,[v\cdot\mathrm{D},v\cdot u]]_+]$	F	ON
284		$i[v\cdot u,[\mathrm{S}\cdot u,[v\cdot\mathrm{D},v\cdot u]]]_+$	F	ON
285		$\epsilon^{\mu\nu\rho\lambda}[u_\mu,[u_\nu,[\mathrm{D}_\rho,u_\lambda]]_+]_+$	F	ON
286		$\epsilon^{\mu\nu\rho\lambda}[\mathrm{D}_\mu,[u_\nu,[u_\rho,u_\lambda]]_+]_+$	F	E
287		$\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot\mathrm{D},[u_\mu,[u_\nu,u_\lambda]]_+]_+$	$\mathrm{D}[d_6]$	E
288		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[u_\nu,[v\cdot\mathrm{D},u_\lambda]]_+]_+$	F	E
289		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[u_\nu,[\mathrm{D}_\lambda,v\cdot u]]_+]_+$	F	ON
290		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[v\cdot u,[u_\nu,u_\lambda]]_+]_+$	$\mathrm{D}[d_{67}]$	ON
291		$\epsilon^{\mu\nu\rho\lambda}v_\rho[v\cdot u,[u_\mu,[u_\nu,\mathrm{D}_\lambda]]_+]_+$	$\mathrm{D}[d_{71}]$	ON
292		$\epsilon^{\mu\nu\rho\lambda}v_\rho[[\mathrm{D}_\mu,u_\nu],[u_\lambda,v\cdot u]_+]_+$	$\mathrm{D}[d_{69}]$	ON
293		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[\mathrm{D}_\nu,[v\cdot u,u_\lambda]]_+]_+$	F	ON
294		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[\mathrm{D}_\nu,[v\cdot u,u_\lambda]]]$	F	ON
295		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[v\cdot u,[u_\nu,\mathrm{D}_\lambda]]_+]_+$	$\mathrm{D}[d_{70}]$	ON
296		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[v\cdot u,[u_\nu,\mathrm{D}_\lambda]]]$	F	ON
297		$\epsilon^{\mu\nu\rho\lambda}v_\rho[u_\mu,[v\cdot u,[u_\nu,\mathrm{D}_\lambda]_+]]_+$	F	ON
298		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[u_\nu,[v\cdot u,u_\lambda]_+]_+]$	F	ON
299		$\epsilon^{\mu\nu\rho\lambda}v_\rho[\mathrm{D}_\mu,[u_\nu,[v\cdot u,u_\lambda]_+]]_+$	F	ON
300		$i[v\cdot u,u_\mu]_+\mathrm{S}\cdot u\mathrm{D}^\mu$+h.c.	$\mathrm{D}[d_{87}]$	E
301		$i[v\cdot u,\mathrm{S}\cdot u]_+u^\mu\mathrm{D}_\mu$+h.c.	$\mathrm{D}[d_{88}]$	E
302		$i[u_\mu,\mathrm{S}\cdot u]_+v\cdot u\mathrm{D}^\mu$+h.c.	$\mathrm{D}[d_{89}]$	E
303		$i[u_\mu,\mathrm{S}\cdot u]_+u^\mu v\cdot\mathrm{D}$+h.c.	$\mathrm{D}[d_{86}]$	E
304		$i[v\cdot u,\mathrm{S}\cdot u]_+v\cdot uv\cdot\mathrm{D}$+ h.c.	$\mathrm{D}[d_{91}]$	E
305		$i[(v\cdot u)^2,[v\cdot u,\mathrm{S}\cdot\mathrm{D}]_+]_+$	$\mathrm{D}[d_{92}]$	ON$'$

Table 2: continued

i	(m,n,p,q)	Terms	F(≡Finite) D(≡Divergent)[d_i]	E ON
306		$i[\mathrm{S}\cdot\mathrm{D},(v\cdot u)^3]_+$	F	E
307		$i[\mathrm{D}_\mu,[v\cdot u,[\mathrm{S}\cdot u,u^\mu]]]_+$	F	E
308		$i[v\cdot u,[\mathrm{D}_\mu,[\mathrm{S}\cdot u,u^\mu]]]_+$	F	E
309		$i[\mathrm{S}\cdot u,[u_\mu,[\mathrm{D}^\mu,v\cdot u]]_+]$	F	ON
310		$i[\mathrm{S}\cdot u,[u_\mu,[\mathrm{D}^\mu,v\cdot u]]]_+$	F	ON
311		$i[u_\mu,[\mathrm{S}\cdot u,[\mathrm{D}^\mu,v\cdot u]]]_+$	F	ON
312		$i[u_\mu,[\mathrm{S}\cdot u,[\mathrm{D}^\mu,v\cdot u]]_+]$	F	ON
313		$i[v\cdot u,[v\cdot u,[v\cdot u,\mathrm{S}\cdot\mathrm{D}]]]_+$	F	ON
314		$i[(v\cdot u)^2,[\mathrm{S}\cdot u,v\cdot\mathrm{D}]_+]_+$	D[d_{90}]	ON
315		$i[[u^\mu,v\cdot u],[\mathrm{S}\cdot\mathrm{D},u_\mu]]_+$	D[d_{80}]	ON
316		$i[u_\mu,\mathrm{S}\cdot u],[\mathrm{D}^\mu,v\cdot u]]_+$	D[d_{79}]	ON
317	(2,0,0,1)	$[\mathrm{S}\cdot\mathrm{D},[v\cdot\mathrm{D},\chi_-]]_+$	F	E
318	(0,2,0,1)	$[\mathrm{S}\cdot u,[v\cdot u,\chi_-]]_+$	F	ON
319	(1,1,1,0)	$i[[v\cdot\mathrm{D},\mathrm{S}\cdot u]_+,\chi_+]_+$	D[d_{117}]	E
320		$i[[\mathrm{S}\cdot\mathrm{D},v\cdot u]_+,\chi_+]_+$	D[d_{118}]	E

Table 3: BChPT counterparts of L.C.-independent HBChPT terms

i	HBChPT term	BChPT counterpart
1	$(\mathcal{V}^\mu\mathcal{A}_\mu)^{m_3}\equiv(v\cdot u)^{j_1}\mathrm{S}\cdot\mathrm{D}(u_\rho\mathrm{D}^\rho)^{j_2}$	$(\frac{i}{\mathrm{m}}u\cdot\mathrm{D})^{j_1}\gamma^5\not{\mathrm{D}}(u^\rho\mathrm{D}_\rho)^{j_2}$
2	$(\mathcal{V}^\nu\mathcal{V}_\nu)^{m_1}\equiv(v\cdot\mathrm{D})^{\mathrm{l}_1}(\mathrm{D}_\nu\mathrm{D}^\nu)^{\mathrm{l}_2}$	$\Big((\mathrm{D}^2+\mathrm{m}^2) -\Big(-(i\not{\mathrm{D}}-\mathrm{m})^2+\frac{i}{8}\sigma^{\mu\nu}[u_\mu,u_\nu]\Big)\Big)^{l_1} \times\Big[(\mathrm{D}_\nu\mathrm{D}^\nu)^{l_2} -\frac{1}{2}\sum\Big[\Big((\mathrm{D}^2+\mathrm{m}^2)-\Big(-(i\not{\mathrm{D}}-\mathrm{m})^2 +\frac{i}{8}\sigma^{\mu\nu}[u_\mu,u_\nu]\Big)\Big)+\mathrm{m}^2\Big] \times(\mathrm{D}^\nu\mathrm{D}_\nu)^{l_2-1}\Big]$
3	$(\mathcal{A}^\alpha\mathcal{A}_\alpha)^{m_2}\equiv(u^\mu u_\mu)^{m_2-1}\mathrm{S}\cdot u$	$(u^\mu u_\mu)^{m_2-1}\gamma^5\not{u}$

Table 4: The 2 L.C.-independent $\mathrm{O}(q^4, \phi^{2n})$ terms whose LECs are fixed relative to an $\mathrm{O}(q^2, \phi^{2n})$ and $\mathrm{O}(q^3)$ terms

	(2,2,0,0) HBChPT Term	BChPT Counterpart
1	$[\mathrm{D}_\mu, [[\mathrm{D}_\nu, u^\mu]_+, u^\nu]_+]_+$	$[\mathrm{D}_\mu, [[\mathrm{D}_\nu, u^\mu]_+, u^\nu]_+]_+$ $+2i\mathrm{m}[[\mathrm{D}_\mu, \not{u}]_+, u^\mu]_+ + 2i\mathrm{m}[\mathrm{D}_\mu, [u^\mu, \not{u}]_+]_+ \equiv \mathrm{O}(q^2)$ or $[\mathrm{D}_\mu, [[\mathrm{D}_\nu, u^\mu]_+, u^\nu]_+]_+$ $+2i\mathrm{m}[[\mathrm{D}_\mu, \not{u}]_+, u^\mu]_+ \equiv \mathrm{O}(q^3)$
2	$[\mathrm{D}_\mu, [[\mathrm{D}_\nu, u^\nu]_+, u^\mu]_+]_+$	$[\mathrm{D}_\mu, [[\mathrm{D}_\nu, u^\nu]_+, u^\mu]_+$ $+2i\mathrm{m}[\mathrm{D}_\mu, [\not{u}, u^\mu]_+]_+ + 2i\mathrm{m}[[\mathrm{D}_\mu, u^\mu]_+, \not{u}]_+ \equiv \mathrm{O}(q^2)$ or $[\mathrm{D}_\mu, [[\mathrm{D}_\nu, u^\nu]_+, u^\mu]_+]_+$ $+2i\mathrm{m}[\mathrm{D}_\mu, [\not{u}, u^\mu]_+]_+ \equiv \mathrm{O}(q^3)$
3	$[\mathrm{D}_\mu, [\mathrm{D}_\nu, [u^\mu, u^\nu]_+]_+]_+$	$[\mathrm{D}_\mu, [\mathrm{D}_\nu, [u^\mu, u^\nu]_+]_+]_+ + 4i\mathrm{m}^0[\mathrm{D}_\mu, [\not{u}, u^\mu]_+]_+ \equiv \mathrm{O}(q^2)$ or $[\mathrm{D}_\mu, [\mathrm{D}_\nu, [u^\mu, u^\nu]_+]_+]_+$ $+2i\mathrm{m}^0[\mathrm{D}_\mu, [\not{u}, u^\mu]_+]_+ \equiv \mathrm{O}(q^3)$

Table 5: BChPT counterparts of L.C.-dependent HBChPT terms

i	HBChPT term	BChPT counterpart
1	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} \quad \prod_{i=1}^{M_1} \mathcal{V}_{\mu_i} \prod_{j=1}^{M_2} \mathcal{A}_{\nu_j}$:	
	(a) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} \; v_{\mu_k} \mathrm{S}_{\nu_l} \prod_{i\neq k}^{N_1} \mathrm{D}_{\mu_i} \prod_{j\neq l}^{N_2} u_{\nu_j}$	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\gamma^5\gamma_{\nu_l}$ $\times \frac{i}{\mathrm{m}} \mathrm{D}_{\mu_k} \prod_{i\neq k}^{N_1} \mathrm{D}_{\mu_i} \prod_{j\neq l}^{N_2} u_{\nu_j}$
	(b) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} \; v_{\mu_k} \prod_{i\neq k}^{M_1} \mathrm{D}_{\mu_i} \prod_{j=1}^{M_2} u_{\nu_j}$	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}$ $\times \frac{i}{\mathrm{m}} \mathrm{D}_{\mu_k} \prod_{i\neq k}^{M_1} \mathrm{D}_{\mu_i} \prod_{j=1}^{M_2} u_{\nu_j}$
	(c) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} \; \prod_{i=1}^{M_1} \mathrm{D}_{\mu_i} \prod_{j=1}^{M_2} u_{\nu_j}$ e.g. $\epsilon^{\mu\nu\rho\lambda}\Big(\mathrm{D}_\mu u_\nu [\mathrm{D}_\rho, \mathrm{D}_\lambda],$ $u_\mu \mathrm{D}_\nu [u_\rho, u_\lambda]\Big)$	$\epsilon^{\mu\nu\rho\lambda}$ $\times\Big(\mathrm{D}_\mu u_\nu [\mathrm{D}_\rho, \mathrm{D}_\lambda] + i\mathrm{m}\gamma_\mu u_\nu [\mathrm{D}_\rho, \mathrm{D}_\lambda],$ $u_\mu \mathrm{D}_\nu [u_\rho, u_\lambda] + i\mathrm{m}\gamma_\nu u_\mu [u_\rho, u_\lambda]\Big)$
	(d) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} \; \mathrm{S}_{\nu_l} \prod_{i=1}^{M_1} \mathrm{D}_{\mu_i} \prod_{j\neq l}^{M_2} u_{\nu_j}$ e.g. $\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_\lambda \Big(\mathrm{D}_\mu [\mathrm{D}_\rho, \mathrm{D}_\nu],$ $u_\mu \mathrm{D}_\nu u_\rho\Big)$	$\epsilon^{\mu\nu\rho\lambda}\gamma^5\gamma_\lambda \mathrm{D}_\mu [\mathrm{D}_\rho, \mathrm{D}_\nu]$ $-i\mathrm{m}\sigma^{\nu\rho}[\mathrm{D}_\rho, \mathrm{D}_\nu],$ $\epsilon^{\mu\nu\rho\lambda}\gamma^5\gamma_\lambda u_\mu \mathrm{D}_\nu u_\rho$ $-i\mathrm{m}\sigma^{\rho\mu}[u_\mu, u_\rho]$

Table 5: continued

i	HBChPT term	BChPT counterpart
2	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}\mathcal{A}_{\nu_j}(v\cdot u)^{j_1}$	
	(a) $\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_{\nu_l}\prod_{i=1}\mathrm{D}_{\mu_i}\prod_{j\neq l}u_{\nu_j}(v\cdot u)$: e.g. : $i\epsilon^{\mu\nu\rho\lambda}\mathrm{S}_{\lambda}\Big(\mathrm{D}_{\nu}u_{\mu}\mathrm{D}_{\rho}v\cdot u+\mathrm{h.c.},$ $[u_{\nu},u_{\rho}]u_{\mu}v\cdot u+\mathrm{h.c.},\ \ \mathrm{D}_{\nu}u_{\mu}\mathrm{D}_{\rho}v\cdot u+\mathrm{h.c.}$	$\epsilon^{\mu\nu\rho\lambda}\gamma^5\gamma_{\lambda}\Big(\mathrm{D}_{\nu}u_{\mu}\mathrm{D}_{\rho}[\mathrm{D}^{\kappa},u_{\kappa}]_{+},$ $[u_{\nu},u_{\rho}]u_{\mu}[\mathrm{D}^{\kappa},u_{\kappa}]_{+}\Big),$ $\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\gamma_{\lambda}\gamma^5\mathrm{D}_{\nu}u_{\mu}\mathrm{D}_{\rho}[\mathrm{D}^{\kappa},u_{\kappa}]_{+}$ $-i\frac{\mathrm{m}}{2}\sigma^{\lambda\rho}[u_{\lambda},\mathrm{D}_{\rho}][\mathrm{D}_{\kappa},u^{\kappa}]_{+}$
	(b) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\ v_{\mu_k}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(v\cdot u)^{j_1}$ up to $\mathrm{O}(q^4)$ sufficient to consider $j_1=1$	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}$ $\times\frac{\mathrm{D}_{\mu_k}}{\mathrm{m}^0}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}[\mathrm{D}_{\kappa},u^{\kappa}]_{+}$
	(c) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{\mu_i}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(v\cdot u)^{j_1}$ $\equiv\mathrm{O}(q^5)$ at least	
	(d) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\ v_{\mu_k}\mathrm{S}_{\nu_l}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j\neq l}^{M_2}u_{\nu_j}(v\cdot u)^{j_1}$ for $\mathrm{O}(q^4)$ terms $j_1=2$: $\epsilon^{\mu\nu\rho\lambda}v_{\mu}\mathrm{S}_{\nu}[\mathrm{D}_{\rho},\mathrm{D}_{\lambda}](v\cdot u)^2$	$\sigma^{\nu\rho}[\mathrm{D}_{\nu},\mathrm{D}_{\rho}][\mathrm{D}_{\kappa},u^{\kappa}]_{+}^{2}$
3	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(\mathrm{S}\cdot\mathrm{D})$	
	(a) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{\mu_i}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(\mathrm{S}\cdot\mathrm{D})$ $\equiv\mathrm{O}(q^5)$ at least	
	(b) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\ v_{\mu_k}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(\mathrm{S}\cdot\mathrm{D})$	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}$ $\times\frac{\mathrm{D}_{\mu_k}}{\mathrm{m}}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}\gamma^5\not{\mathrm{D}}$
4	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}\mathcal{A}_{\nu_j}(u^{\rho}\mathrm{D}_{\rho})^{j_2}(v\cdot u)^{j_1}$ $\equiv$ at least of $\mathrm{O}(q^5)$	

Table 5: continued

i	HBChPT term	BChPT counterpart
5	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(u^\rho \mathrm{D}_\rho)^{j_2}(\mathrm{S}\cdot\mathrm{D})$ $\equiv$ at least of $\mathrm{O}(q^6)$	
6	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(\mathrm{S}\cdot\mathrm{D})(v\cdot u)^{j_1}$ $\equiv$ at least of $\mathrm{O}(q^5)$	
7	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(\mathrm{S}\cdot\mathrm{D})(u_\rho \mathrm{D}^\rho)^{j_2}(v\cdot u)^{j_1}$ $\equiv$ at least of $\mathrm{O}(q^7)$	
8	$\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathcal{V}_{\mu_i}\prod_{j=1}^{M_2}\mathcal{A}_{\nu_j}(u^\rho \mathrm{D}_\rho)^{j_2}:$	
	(a) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\prod_{i=1}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(u^\rho \mathrm{D}_\rho)^{j_2}$ $\equiv \mathrm{O}(q^6)$ at least	
	(b) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}v_{\mu_k}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j=1}^{M_2}u_{\nu_j}(u_\rho \mathrm{D}^\rho)^{j_2}$ $\equiv \mathrm{O}(q^5)$ at least	
	(c) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\ \mathrm{S}_{\nu_l}\prod_{i=1}^{M_1}\mathrm{D}_{\mu_i}\prod_{j\neq l}^{M_2}u_{\nu_j}(u_\rho \mathrm{D}^\rho)^{j_2}$ $\equiv \mathrm{O}(q^5)$ at least	
	(d) $\epsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\mathrm{S}_{\nu_l}v_{\mu_k}\prod_{i\neq k}^{M_1}\mathrm{D}_{\mu_i}\prod_{j\neq l}u_{\nu_j}(u^\rho \mathrm{D}_\rho)^{j_2}$ up to $\mathrm{O}(q^4)$ $j_2=1$: e.g $i\epsilon^{\mu\nu\rho\lambda}v_\nu \mathrm{S}_\mu u^\kappa u_\rho \mathrm{D}_\lambda \mathrm{D}_\kappa + \mathrm{h.c.}$	$\frac{\epsilon^{\mu\nu\rho\lambda}}{\mathrm{m}}\gamma^5\gamma_\mu \mathrm{D}_\nu u^\kappa u_\rho \mathrm{D}_\lambda \mathrm{D}_\kappa$ $+2i\sigma^{\rho\lambda}[u^\kappa u_\rho, \mathrm{D}_\lambda]\mathrm{D}_\kappa$

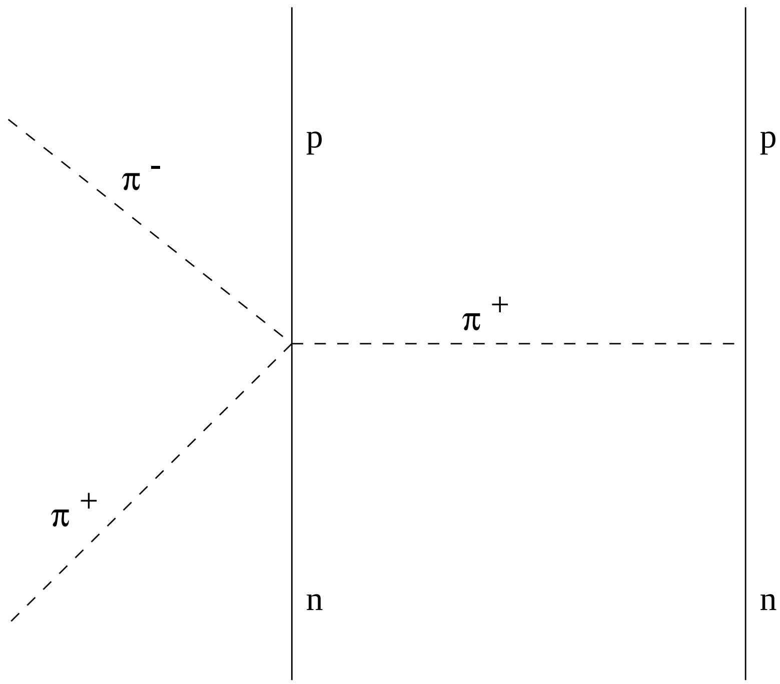

Figure 1: “Contact graph” of pion DCX

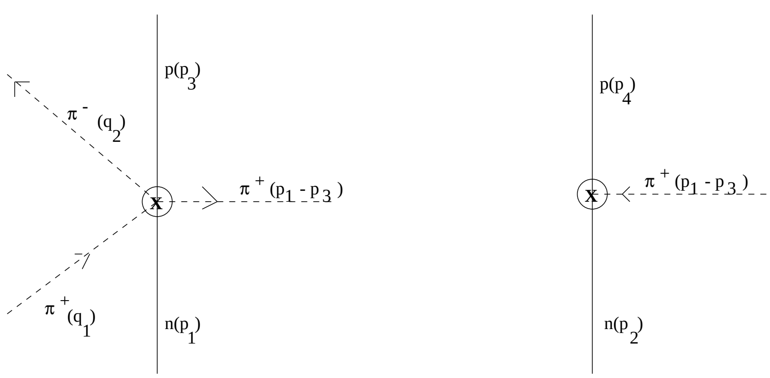

Figure 2: O(q^4) operator insertion (indicated by a cross in a circle) for $\bar{p}(\pi^+)^2\pi^- n$- and $\bar{p}\pi^+ n$-vertices that figure in the “contact graph” of pion DCX